A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

March 2007

EPL, **77** (2007) 50010
doi: 10.1209/0295-5075/77/50010

www.epljournal.org

# Achieving synchronization of networks by an auxiliary hub

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received 16 October 2006; accepted in final form 23 January 2007
published online 26 February 2007

PACS 05.45.Xt – Synchronization; coupled oscillators
PACS 89.75.-k – Complex systems
PACS 87.18.Sn – Neural networks

**Abstract** – In this letter, we study complete synchronization in quite generic networks of coupled oscillators, where both the connectivity topology and coupling mechanism may be arbitrary. We propose a method, *i.e.*, introducing an auxiliary hub which decreases the average path length and meanwhile increases the clustering coefficient of the network, to achieve complete synchronization. We demonstrate that the method is successful in synchronizing some classical network models, which cannot synchronize intrinsically. That is, a network, which itself is impossible to synchronize, can adaptively achieve the complete synchronization by introducing an auxiliary hub. The present results give insights on how the network structure influences the synchronizability and how the synchronization of generic networks can be achieved regardless of their own complexity.



Recently, there is a highly focused interest in networks of coupled oscillators, which constitute many models in nature. One of the most striking phenomena in these models is the spontaneous synchronization, which can be observed in various types of systems, such as living and nonliving oscillators at every scale ranging from the nucleus to the cosmos [1–4]. On the other hand, it has been shown that most of the real networks have very distinguished connectivity structures, such as small-world property and scale-free structure, etc. [5–9]. The investigation of the dynamics of such complex networks is and will be a big challenge in current and future research. Mainly two questions have to be addressed. First, how does the connectivity topology in networks influence synchronizability? Second, which mechanisms can achieve the synchronization irrespective of the individual topologies in complex networks? The first question concerning the influence of the topology has been intensively investigated over the past years [10–14]. Other studies focused on the question how to enhance the complete synchronization by choosing the coupling mechanism but not altering the connectivity topology in networks, such as weighted networks [15–17], adaptive networks [18,19], etc. Anyway, more evidences show that the characteristics of complex networks, *e.g.*, the smaller average path length, may enhance the synchronizability.

In this letter, we address the second question by asking how to achieve the complete synchronization of network dynamics regardless of the intrinsic complexity of networks. The network considered here is quite generic in the sense that the connectivity structure and the coupling mechanism may be arbitrary. For generic networks of oscillators, the complete synchronization is in general difficult to achieve due to the dynamics of individual oscillators, the form of coupling, or the complexity of connectivity topology. As we know, an ideal hub will dramatically decrease the average path length, and meanwhile increase the clustering coefficient of networks. In nature, hubs are quite popular in complex networks, and may arise as an example by preferential attachment in the evolution of scale-free networks. Therefore, and motivated by the idea that the smaller average path length may enhance the synchronizability, we demonstrate in this letter that complete synchronization can be achieved by introducing an auxiliary hub without altering the connectivity topology or the coupling mechanism in the pristine networks.

We start by considering a generic network of $N$ identical oscillators

$$\dot{X}_i = F(X_i) + H_i(X_1, \cdots, X_N), \quad i = 1, 2, \cdots, N, \qquad (1)$$

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where $F(X)$ governs the dynamics of individual oscillators, and $X_i = (x_{i1}, \cdots, x_{im}) \in \mathrm{R}^m$ denotes $m$ state variables of the $i$-th node. The functions $H_i, i = 1, 2, \cdots, N$, constitute the network structure, which is supposed to have arbitrary connectivity topologies and coupling mechanisms. We assume $H_i(X, \cdots, X) = H_j(X, \cdots, X)$ for all $i, j = 1, 2, \cdots, N$ and any $X \in \mathrm{R}^m$, so that the synchronous manifold $M = \{X_1 = X_2 = \cdots = X_N\}$ is invariant in system (1). Generally, it is very difficult or impossible to achieve the synchronization in such networks due to the complexity of the node's dynamics, *i.e.*, $F(X)$, or the connectivity structure and coupling mechanism, *i.e.*, $H_i(X_1, \cdots, X_N)$. Here, our goal is to investigate the complete synchronization by introducing an auxiliary hub into the network (1). Denoting the state variables of the hub by $Y = (y_1, y_2, \cdots, y_m)$, we obtain a new network of $(N+1)$ coupled oscillators

$$\dot{X}_i = F(X_i) + H_i(X_1, \cdots, X_N) + U_i(X_i, Y), \quad i = 1, 2, \cdots, N, \tag{2a}$$

$$\dot{Y} = F(Y), \tag{2b}$$

where $U_i(X_i, Y)$ is the coupling from the $i$-th node to the hub $Y$. We let $U_i(X, X) \equiv 0$ for all $i = 1, 2, \cdots, N$ and any $X \in \mathrm{R}^m$, which implies that the addition of the hub is only auxiliary for the complete synchronization of the original network (1). Also note that such hub is unidirectionally coupled with other nodes, and hence is different from the generic hub in networks. For convenience, we refer to the old network (1) as the pristine network. We address the problem: Is it possible to achieve the synchronization in the pristine network (1) by such an auxiliary hub with a certain coupling $U_i$ in (2)?

For clarity, we consider generic networks of diffusively coupled oscillators with the common output, *i.e.*,

$$\dot{X}_i = F(X_i) + \sum_{j=1}^{N} g_{ij}(H(X_j) - H(X_i)), \quad i = 1, 2, \cdots, N, \tag{3}$$

where $H(X)$ is the output function of individual oscillators, and $G = (g_{ij})$ is a generic coupling matrix. The matrix $G$ may be divided into two matrices, *i.e.*, $G = C \otimes W$ defined by $g_{ij} = c_{ij} w_{ij}$. The matrix $C = (c_{ij})$ denotes the connectivity matrix of networks, *i.e.*, $c_{ij} = 1$, if there is a connection between the $i$-th node and the $j$-th node, $c_{ij} = 0$, otherwise. The matrix $W = (w_{ij})$ is the matrix of weighted coupling, and $w_{ij}$ denotes the coupling strength between the $i$-th node and the $j$-th node, where $w_{ij} = 0$ if $c_{ij} = 0$. So the matrix $C$ is the binary adjacency matrix showing the complexity of network structure, and it may be regular, random, or the connectivity matrix constituting small-world or scale-free networks. The coupling may be weighted or unweighted (*i.e.*, $w_{ij} \equiv \sigma$, if $w_{ij} \neq 0$), and linear or nonlinear (*i.e.*, the coupling function $H(X)$ is nonlinear). To realize the complete synchronization in the pristine network (3) regardless of its intrinsic complexity, we introduce an auxiliary hub with adaptive unweighted vector coupling, and obtain a new network of oscillators

$$\dot{X}_i = F(X_i) + \sum_{j=1}^{N} g_{ij}(H(X_j) - H(X_i)) + k(Y - X_i), \quad i = 1, 2, \cdots, N, \tag{4a}$$

$$\dot{Y} = F(Y) \tag{4b}$$

with an adaptive coupling strength

$$\dot{k} = \gamma \sum_{i=1}^{N} \| Y - X_i \|^2, \tag{4c}$$

where $\gamma$ is an arbitrary positive constant, and $\| Y - X_i \|^2 = \sum_{l=1}^{m} (y_l - x_{il})^2$. To guarantee the coupling is sufficiently weak, we generally let $0 < \gamma \ll 1$. The introduction of the variable coupling (4$c$) is to adaptively find the suitable coupling strength, regardless of the complexity of the pristine networks. Such adaptive law may be replaced with other forms, *e.g.*, replacing $\sum_{i=1}^{N} \| Y - X_i \|^2$ in (4$c$) by $\sum_{i=1}^{N} \| Y - X_i \|^2 / (1 + \sum_{i=1}^{N} \| Y - X_i \|^2)$. Similar to the results in [19,20], some necessary conditions, *e.g.*, the uniform Lipschitz condition on the functions $F(X)$ and $H(X)$, are needed to guarantee the stable synchronization motion in (4). Also note that the partial variables (not complete vector) coupling with the hub is sufficient in some cases, see examples below.

Now we consider a concrete network, *i.e.*, the classic network model of $x$-coupled Rössler oscillators. In the model, the state variables of the individual oscillators $X = (x, y, z)$, the dynamics function $F(X) = (-y - z, x + 0.2y, 0.2 + (x - 7)z)$, and the coupling (*i.e.* output) function $H(X) = (x, 0, 0)$. This network consists of a ring of $N$ nodes each coupled to its $2n$ nearest neighbors with the overall strength $\sigma$, *i.e.*, the coupling matrix is in the form of $G = \sigma C$, where the adjacency matrix $C = (c_{ij})$ is a circulant matrix with 1 on $2n$ (circulantly) adjacent diagonals, 0 otherwise. The network dynamics is governed by

$$\begin{aligned} \dot{x}_i &= -y_i - z_i + \sigma \sum_{j=1}^{N} c_{ij}(x_j - x_i), \\ \dot{y}_i &= x_i + 0.2 y_i, \\ \dot{z}_i &= 0.2 + (x_i - 7) z_i, \end{aligned} \tag{5}$$

This model has been extensively investigated by the method of the master stability function in refs. [11,21]. It is well known that such network is difficult to synchronize due to the short-wavelength bifurcation. Especially, the overall coupling strength $\sigma$ is crucial for the synchronization of the network, and the complete synchronization can be achieved only in a finite interval of $\sigma$. On the other hand, as the size of the network is increased, the effective interval of $\sigma$ for the synchronization is smaller and smaller, and eventually approaches zero. This indicates that there exists an upper limit on the number of coupled oscillators for synchronizing the network. In this letter, we let $\sigma = 5$ be a fixed value, so that the pristine network cannot synchronize even for the smallest number of nodes (*i.e.*, $N = 2$). To achieve the complete synchronization of such pristine network, which is itself impossible to synchronize, we introduce an auxiliary hub $X = (x, y, z)$ to give a new network with adaptive $y$-coupling

$$\begin{aligned} \dot{x}_i &= -y_i - z_i + \sigma \sum_{j=1}^{N} c_{ij}(x_j - x_i), \\ \dot{y}_i &= x_i + 0.2y_i + k(y - y_i), \\ \dot{z}_i &= 0.2 + (x_i - 7)z_i, \end{aligned} \tag{6a}$$

$$\dot{x} = -y - z, \quad \dot{y} = x + 0.2y, \quad \dot{z} = 0.2 + (x - 7)z \tag{6b}$$

$$\dot{k} = \gamma \sum_{i=1}^{N} (y - y_i)^2, \tag{6c}$$

where $i = 1, 2, \cdots, N$. Numerical results show that the pristine network is synchronizable with such an auxiliary hub. Figure 1 shows the temporal evolution of the absolute synchronization error $E$ defined by $E = \sum_{i=1}^{N} [|x_i - x| + |y_i - y| + |z_i - z|]$ and the adaptive coupling strength $k$, where the pristine network structure is the simple cycle configuration, *i.e.*, $n = 1$. Further, we investigate the influence of the connectivity structure on the synchronizability in this network. Note that as the average degree of the pristine network, $d = 2n$, is increased (*i.e.*, $n$ is increased), it was shown in [11] that the upper limit of the number of nodes for the synchronization is increased. This indicates that increasing the average degree enhances the synchronizability of the network dynamics (5). However, in the present consideration the overall strength $\sigma$ is fixed as 5, which exceeds the finite interval of $\sigma$ for the synchronization. So we speculate that in such case the increase of $n$ will be a negative contribution to the complete synchronization of the pristine network (5). This can be certified by investigating the network with an auxiliary hub, *i.e.*, system (6). By numerically stimulating the network (6) with the different values of $n$, we find that the converged coupling strength, $k_c$ (*i.e.*, $k \to k_c$), increases as $n$ increases, see fig. 2. We also find that the addition of a few shortcuts in the pristine network will enhance the synchronizability. However, when too many random shortcuts are added to the pristine network the converged coupling strength in the network (6) will increase. This implies that with a bigger overall coupling strength $\sigma$ (here $\sigma = 5$) the addition of too many random shortcuts weakens the synchronizability in the pristine network.

Next we consider the case that the introduced hub is weighted coupled with the pristine network. Simply, we realize such a coupling through replacing the coupling control $U_i = k(Y - X_i)$ and the adaptive law $\dot{k} = \gamma \sum_{i=1}^{N} \parallel Y - X_i \parallel^2$ in system (4) by $U_i = k_i(Y - X_i)$ and $\dot{k}_i = \gamma \parallel Y - X_i \parallel^2$, respectively. Applying this coupling

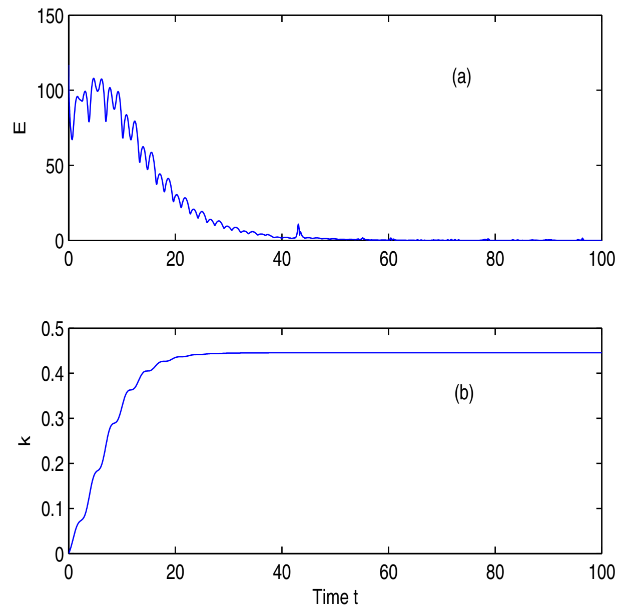


Fig. 1: (a) and (b) show the temporal evolution of the absolute synchronization error $E = \sum_{i=1}^{N} [|x_i - x| + |y_i - y| + |z_i - z|]$ and the adaptive coupling strength $k$ in system (6), respectively, where the pristine network structure is the simple cycle configuration, *i.e.*, $n = 1$.

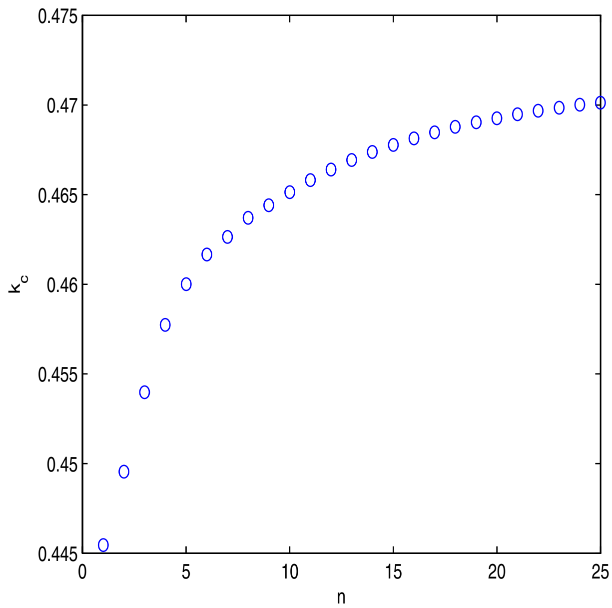


Fig. 2: The relation between the converged value $k_c$ of coupling strength $k$ in system (6) and the average degree $d = 2n$ of its pristine network, which shows that $k_c$ increases as $n$ increases.

scheme to the network of $x$-coupled Rössler oscillators, gives

$$\begin{aligned} \dot{x}_i &= -y_i - z_i + \sigma \sum_{j=1}^{N} c_{ij}(x_j - x_i), \\ \dot{y}_i &= x_i + 0.2y_i + k_i(y - y_i), \\ \dot{z}_i &= 0.2 + (x_i - 7)z_i, \end{aligned} \tag{7a}$$

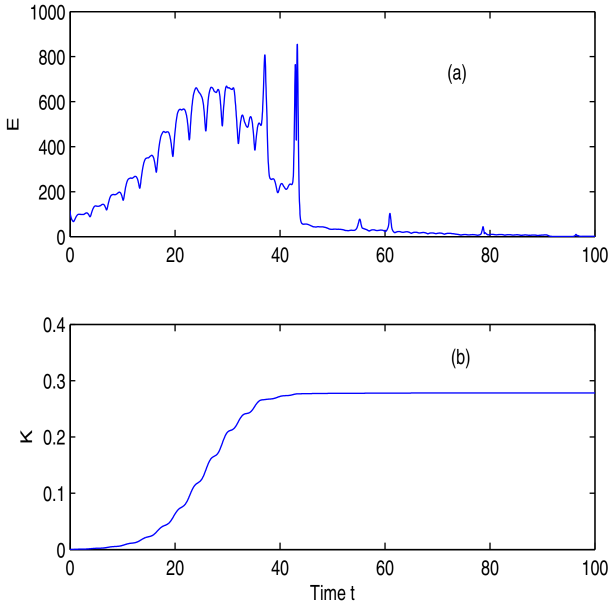


Fig. 3: (a) and (b) show the temporal evolution of the absolute synchronization error $E$ and the average coupling strength $K=\frac{1}{N}\sum_{i=1}^{N}k_i$ in the weighted coupling system (7), respectively, where the connectivity structure in its pristine network is same as that in fig. 1.

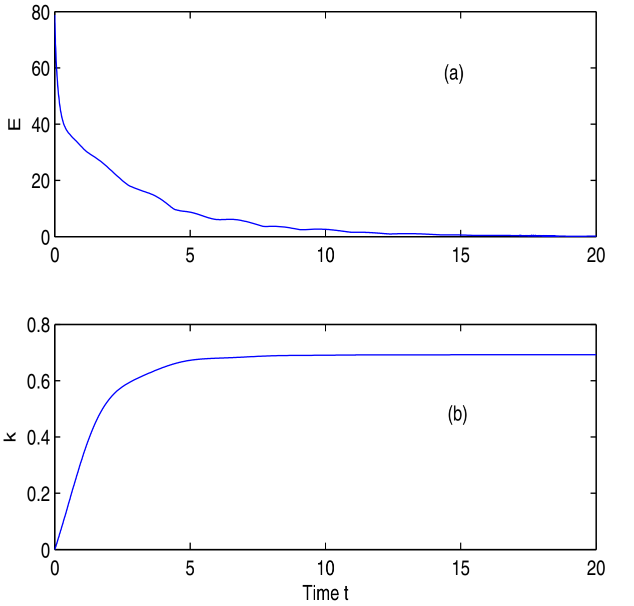


Fig. 4: (a) and (b) show the temporal evolution of the absolute synchronization error $E$ and the coupling strength $k$ in the network (8) with nonlinearly coupled oscillators, respectively.

$$\dot{x}=-y-z, \quad \dot{y}=x+0.2y, \quad \dot{z}=0.2+(x-7)z \tag{7b}$$

$$\dot{k}_i=\gamma(y-y_i)^2, \tag{7c}$$

where $i=1,2,\cdots,N$. The synchronizability of such network is shown in fig. 3, where the pristine network is chosen as the simple cycle configuration (*i.e.*, $n=1$). To compare with the unweighted coupling, we normalize the adaptive law of the coupling strength, *i.e.*, replacing $\sum_{i=1}^{N}(y-y_i)^2$ in (6$c$) by $\frac{1}{N}\sum_{i=1}^{N}(y-y_i)^2$. We find numerically that for almost all initial values the converged value of the average coupling strength $K=\frac{1}{N}\sum_{i=1}^{N}k_i$ in the weighted coupling is smaller than the converged strength of $k$ in the unweighted one. A similar result is also found in the case that $\sum_{i=1}^{N}(y-y_i)^2$ in (6$c$) and $(y-y_i)^2$ in (7$c$) are replaced by $\sum_{i=1}^{N}(y-y_i)^2/[N+\sum_{i=1}^{N}(y-y_i)^2]$ and $(y-y_i)^2/[1+(y-y_i)^2]$, respectively.

(*Remark*: In all numerical stimulations above we let $N=100$, $\gamma=0.001$, the initial-state values be same, which are randomly chosen, and the initial coupling strength $k$ (or $k_i$) be always zero.)

Finally, we consider the case that the pristine network is nonlinearly coupled, *i.e.*, the coupling function $H(X)$ in (3) is nonlinear. Such mechanism of nonlinear coupling is universal in nature, *e.g.*, pulse coupling in biological networks. The numerical evidence of synchronization in such networks has not yet been studied in its full details. Here we only give a case study. Again we use the model of $x$-coupled Rössler oscillators with $H(X)=(e^x,0,0)$ to illustrate that the present synchronization scheme based on the auxiliary hub is still effective. For the case of the simple cycle configuration the network with an auxiliary hub is given by

$$\begin{aligned}
\dot{x}_i &= -y_i-z_i+\sigma(e^{x_{i+1}}-2e^{x_i}+e^{x_{i-1}}),\\
\dot{y}_i &= x_i+0.2y_i+k(y-y_i),\\
\dot{z}_i &= 0.2+(x_i-7)z_i,
\end{aligned} \tag{8a}$$

$$\dot{x}=-y-z, \quad \dot{y}=x+0.2y, \quad \dot{z}=0.2+(x-7)z \tag{8b}$$

$$\dot{k}=\gamma\sum_{i=1}^{N}(y-y_i)^2, \tag{8c}$$

where $i=1,2,\cdots,N$, $x_0=x_N$, and $x_{N+1}=x_1$. The responding numerical results are shown in fig. 4, where $N=100$ and $\gamma=0.04$.

These examples above show that the addition of an auxiliary hub can achieve the complete synchronization of a network, which is itself impossible to synchronize. The efficiency of such synchronization is involved to the dynamics of nodes, *i.e.*, the function $F(X)$, the coupling function $H(X)$, and the coupling matrix $G$ in the pristine network (3). For example, when we replace the coupling function $H(X)=(e^x,0,0)$ in system (8) by $H(X)=(x^2,0,0)$, the network cannot synchronize. As shown above, in particular, the topology structure in the pristine network influences such synchronization scheme. So the prior knowledge about the pristine network is helpful to perform the synchronization more effectively. For example, for those networks with heterogeneous distribution it is probably not necessary to couple the auxiliary hub with all nodes. These problems remain to study further.

In conclusion, from the perspective of physics the addition of such an auxiliary hub is equivalent to giving a common driving to each node in the networks. Therefore, the present results indicate that a network, regardless of its own complexity, may be synchronized by a certain control, *e.g.*, $k(Y - X_i)$ in (4), which gives an insight into how to control the complete synchronization of complex networks. This also agrees with the phenomenon of neural biology: A common stimulus (*e.g.*, the visual stimulus) may drive the neural networks to produce the coordinative behavior although the intrinsic mechanism of the networks is very complex. Thus, the present results have potential applications to explore the dynamics mechanism of the collective behavior in the neural networks, an important neuronal activity [22]. In particular, the networks in the form of (4) combining the linear and nonlinear couplings may be used to investigate the complementary role of electrical and chemical synapses in the synchronization of interneuronal networks [23], where linear couplings represent gap junctions, and nonlinear couplings represent excitatory or inhibitory synapses. Recent studies on complex brain dynamics in [24] enhance the possibility of the idea above, *i.e.*, the extension of the proposed scheme to investigate the relevance of synchronization in artificial or natural systems and clustered synchronization in neural systems.

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The authors would like to thank the anonymous referees for their valuable suggestions. This work is supported by the National Natural Science Foundation (10572080; 10432010), Shanghai Rising-Star Program (05QMX1422), Dawn Project of the Education Committee of Shanghai Municipality (05SG41; 04YQHB089), the Hertie Foundation, and the Alexander von Humboldt Foundation.

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