FIAS Frankfurt Institute for Advanced Studies

FIAS Summer School

THEORETICAL NEUROSCIENCE & COMPLEX SYSTEMS

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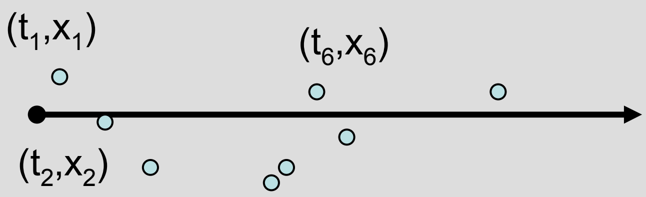
Outline

1. Concepts in time series Analysis
2. Basic Statistics
3. Auto Correlation
4. Auto Regressive Modeling
5. Fourier Transformation
6. Wavelet Transformation

- A *time series* is a set of data pairs

$$(t_a, x_a), a = 1,2,3,\ldots, N$$



- *t* is the time, *x* is the data value
- *x* can be anything, e.g. voltage, height, temperature …

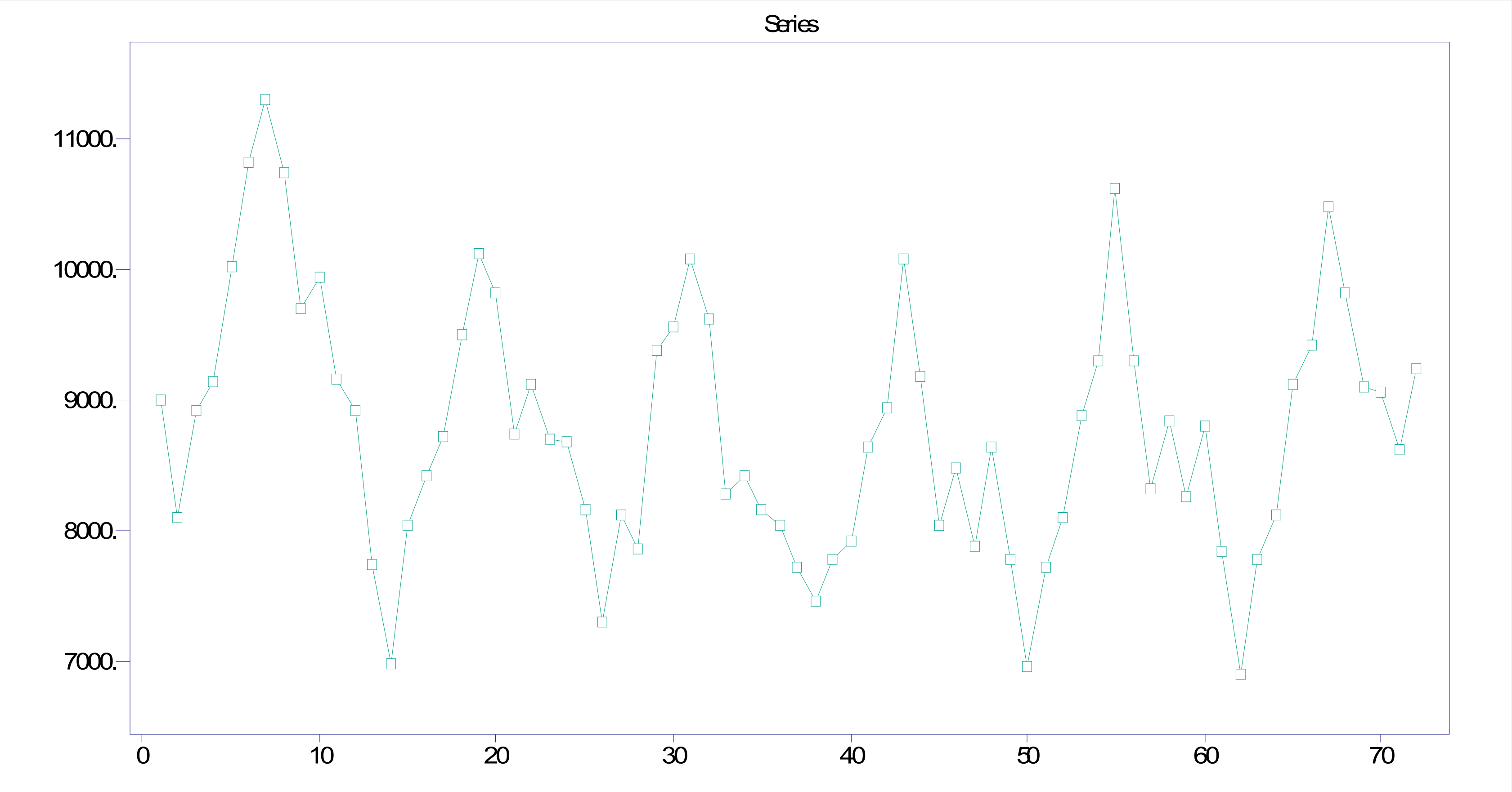
- Usually, times are assumed error-free → Data = Signal + Error

$$x(t) = f(t) + \varepsilon$$

1. ***Visual Inspection* !!**
   - Plot *x* as a function of *t and* Explore
2. **Properties of interest**
   - Basic Properties like mean and std
   - Trends
   - periodic components /limit cycles
   - Auto correlation
3. **Explore data with sophisticated Analysis techniques**

Accidental Deaths in U.S.A. (DEATHS.TSM)

- Monthly totals: January 1973 to December 1978.
- Strong seasonal pattern: high in July, low in Feb.



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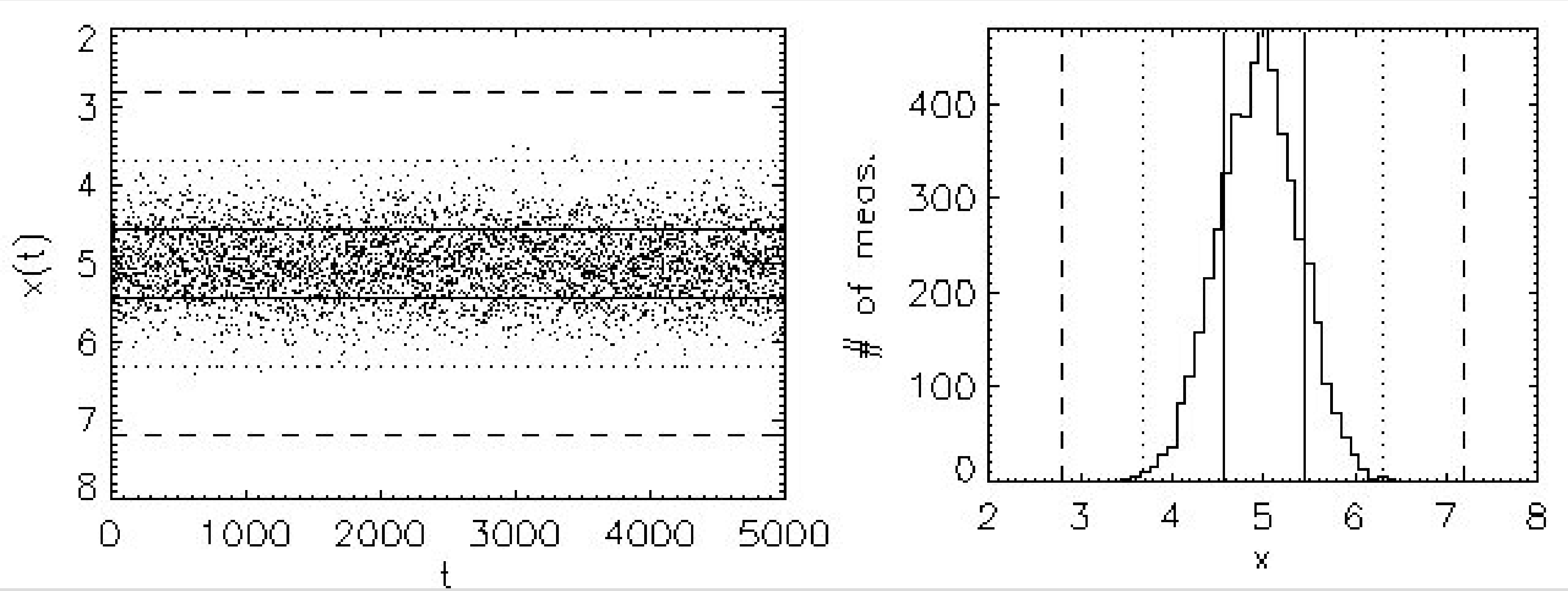
Real underlying process

- μ : Mean = expected value
- σ : Standard deviation = expected variation from mean

Estimation

- $\hat{\mu}$: Average = estimated
- $\hat{\sigma}$: Sample standard deviation



Real underlying process

- μ : Mean = expected value
- σ : Standard deviation = expected variation from mean

Estimation

- $\hat{\mu}$: Average = estimated
- $\hat{\sigma}$: Sample standard deviation d

$$\bar{x} = \frac{1}{N}\sum x$$

$$s = \sqrt{\frac{1}{N-1}\sum (x-\bar{x})^2}$$

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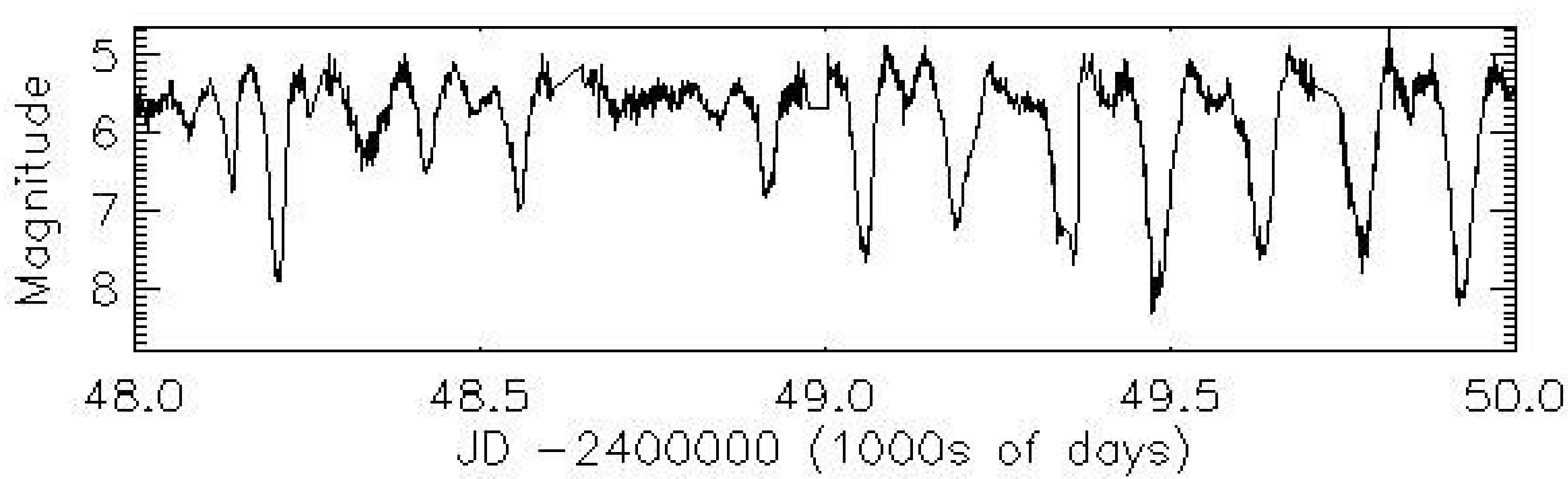
Autocorrelation

- For a range of “periods” (τ), compare each data point x(t) to a point x(t+τ)

- The value of the correlation function at each τ is a function of the average difference between the points

Example Autocorrelation



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Stationary signals:

For a stationary signal differences in time a just dependent on a lag and independent of t itself:

Ergodic signals:

For a ergodic signal the population statistics is the same a statistics over time

→ an ergodic signal must visit all potentially existing points in the phase space in an infinite amount off time

→Attractor states or limit cycle does not yield time series that are ergodic !!!

Auto Regressive Moving Average (ARMA) processes, are an important class of linear time series models. They provide a flexible parametric structure to approximate the behavior of stationary processes, and lead to a prediction theory that is relatively simple and elegant.

Without loss of generality, we assume $\{X_t\}$ has zero mean, since if $\{Y_t\}$ has mean μ, $X_t = Y_t - \mu$ has mean zero.

The AR Process

One of the most intuitive ways to model the behavior of a time series, is to regress X_t on its past p values, say. The resulting model is called an AutoRegression of order p, or AR(p):

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + Z_t, \qquad \{Z_t\} \sim WN(0,\sigma^2).$$

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December, 21, 1807

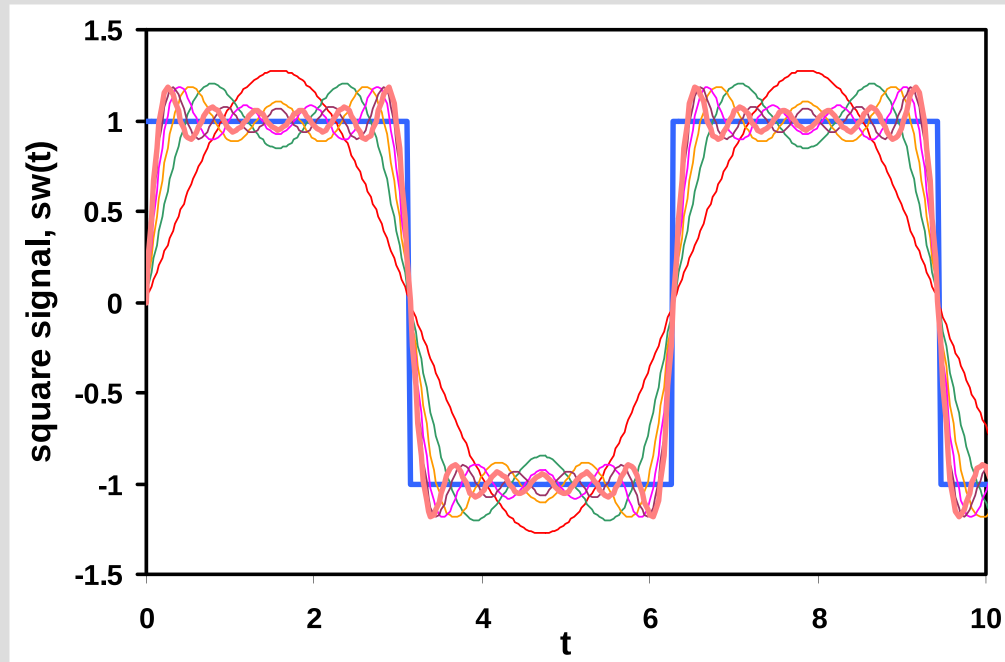
Jean B. Joseph Fourier
(1768-1830)

"An arbitrary function, continuous or with discontinuities, defined in a finite interval by an arbitrarily capricious graph can always be expressed as a sum of sinusoids"

J.B.J. Fourier

Square wave reconstruction from spectral terms

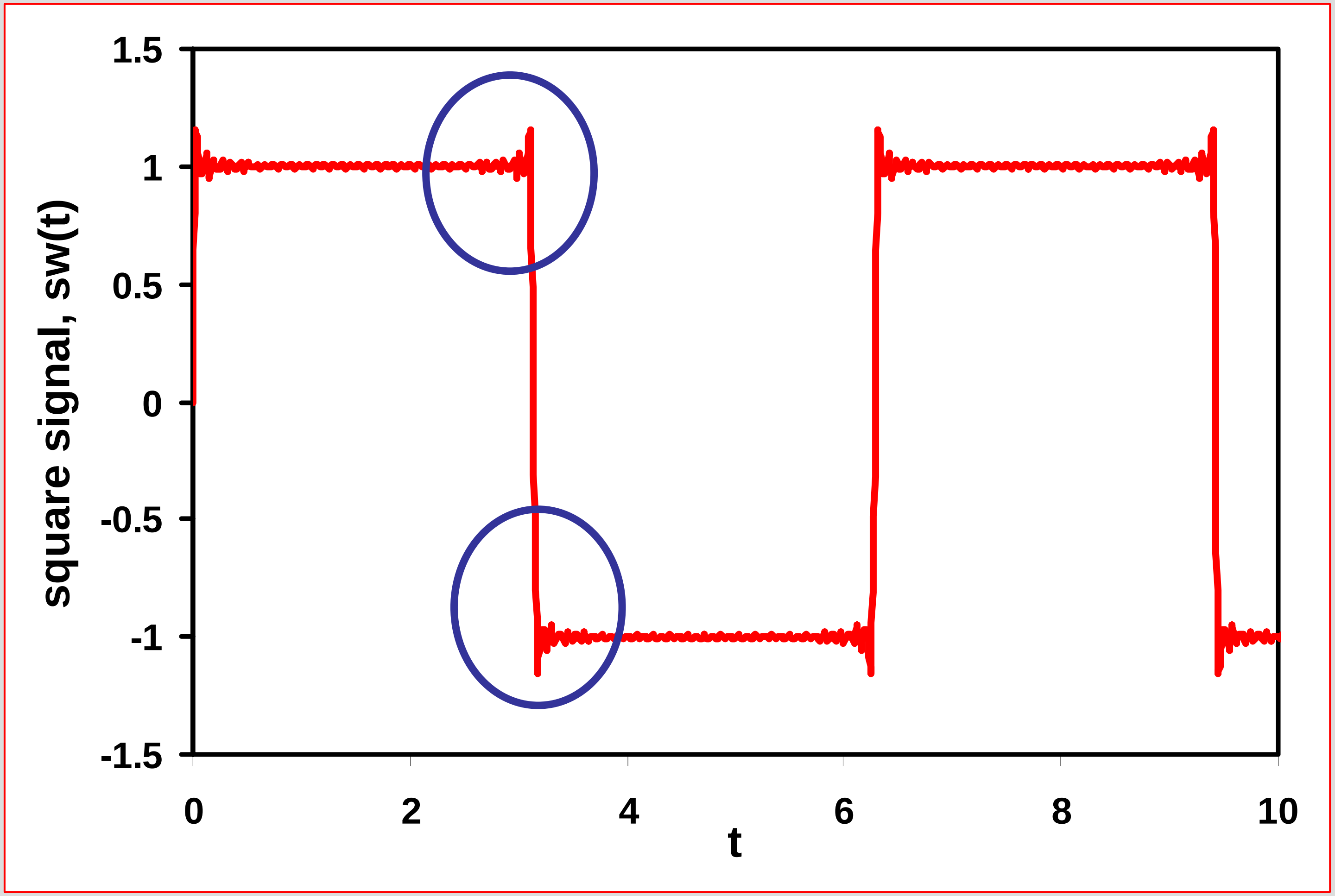
$$sw(t) = \sum_{k=1}^{11} \left[b_k \cdot \sin(kt) \right]$$



Convergence may be slow (~1/k) - ideally need infinite terms.

Overshoot exist at each discontinuity

$$sw_{79}(t) = \sum_{k=1}^{79} \left[-b_k \cdot \sin(kt) \right]$$



- Complex function representation through simple building blocks
  - Basis functions

$$\text{Complex Function} = \sum_i (\text{weight})_i \bullet (\text{Simple Function})_i$$

- Using sinusoids as building blocks ➔ Fourier transform
  - Frequency domain representation of the function

$$F(\omega) = \int f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi} \int F(\omega) e^{j\omega t} d\omega$$

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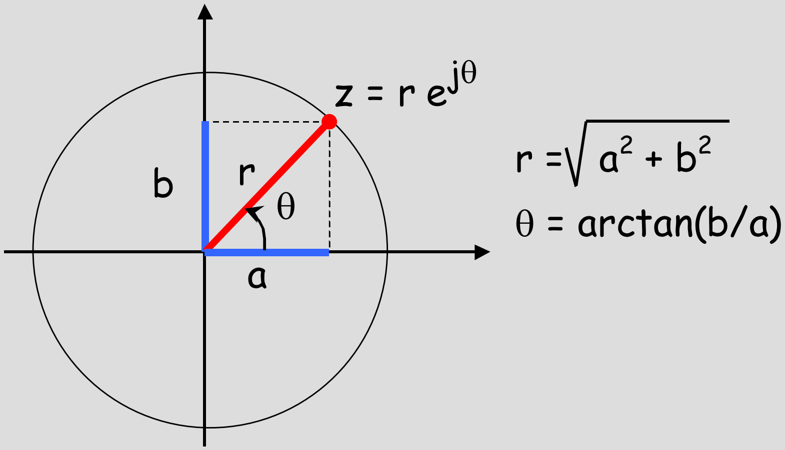
- Recall that FT uses complex exponentials (sinusoids) as building blocks.

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$F(\omega) = \int f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi}\int F(\omega) e^{j\omega t} d\omega$$

- For each frequency of complex exponential, the sinusoid at that frequency is compared to the signal.



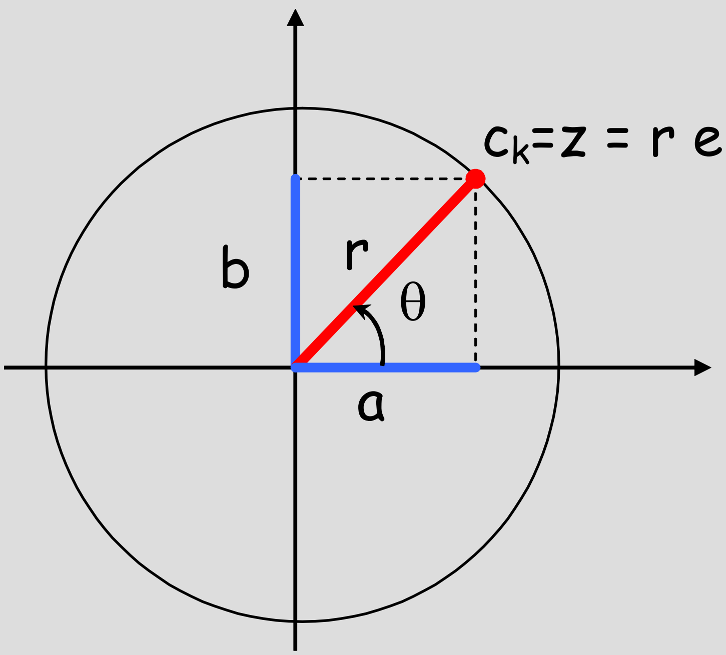
- If the signal consists of that frequency, the correlation is high → large FT coefficients.

Euler's notation:

$e^{-it} = (e^{it})^* = \cos(t) - i \cdot \sin(t)$

$$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-ik\omega t} dt$$

$$c_k = r\left(\cos(\omega t) - i \sin(\omega t)\right)$$



$c_k = z = r\, e^{i\theta}$

$r = \sqrt{a^2 + b^2}$

$\theta = \arctan(b/a)$

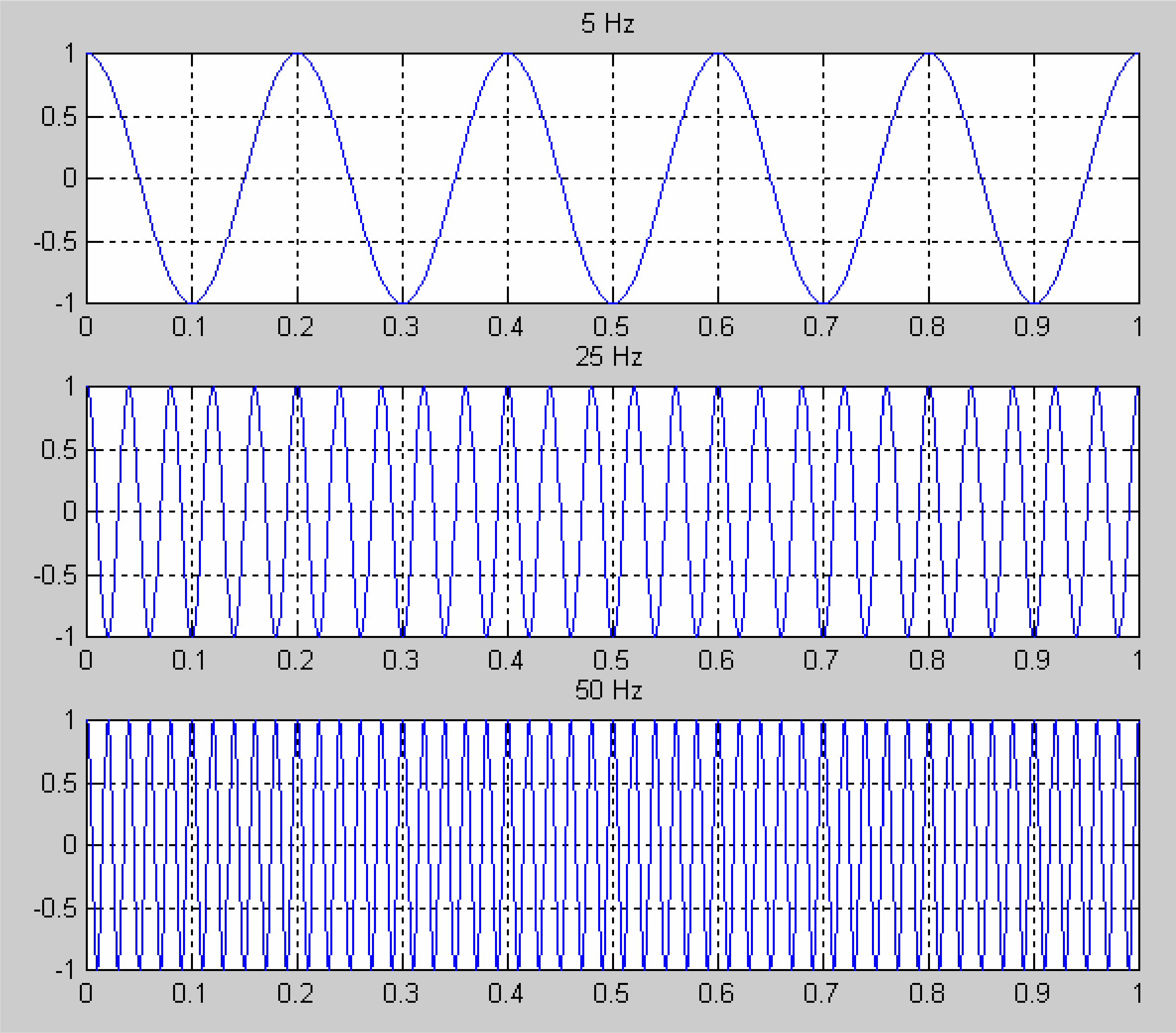
IF s(t) is real → Note: $c_{-k} = (c_k)^*$

Example

$$x_1(t) = \cos(2\pi \cdot 5 \cdot t)$$

$$x_2(t) = \cos(2\pi \cdot 25 \cdot t)$$

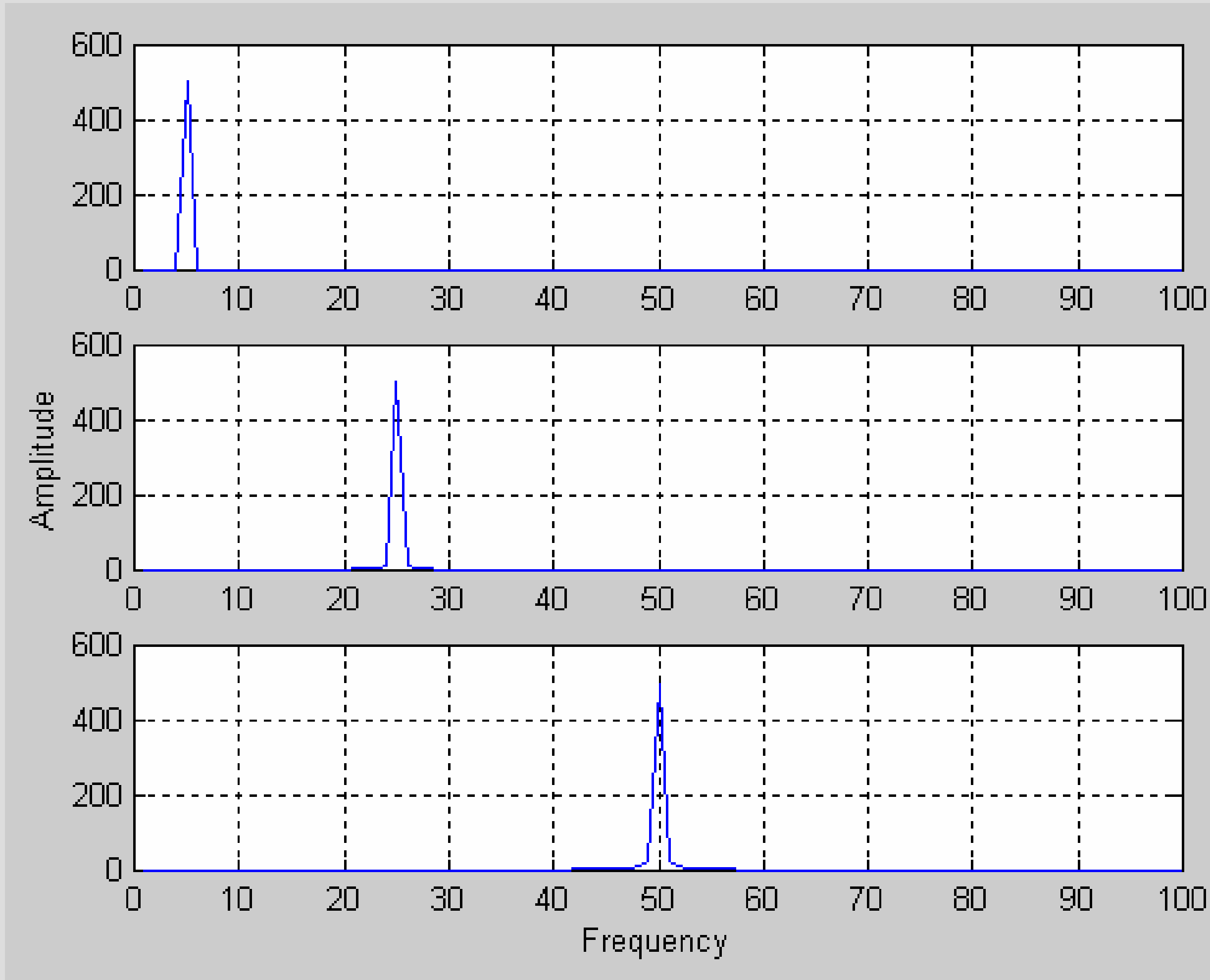
$$x_3(t) = \cos(2\pi \cdot 50 \cdot t)$$



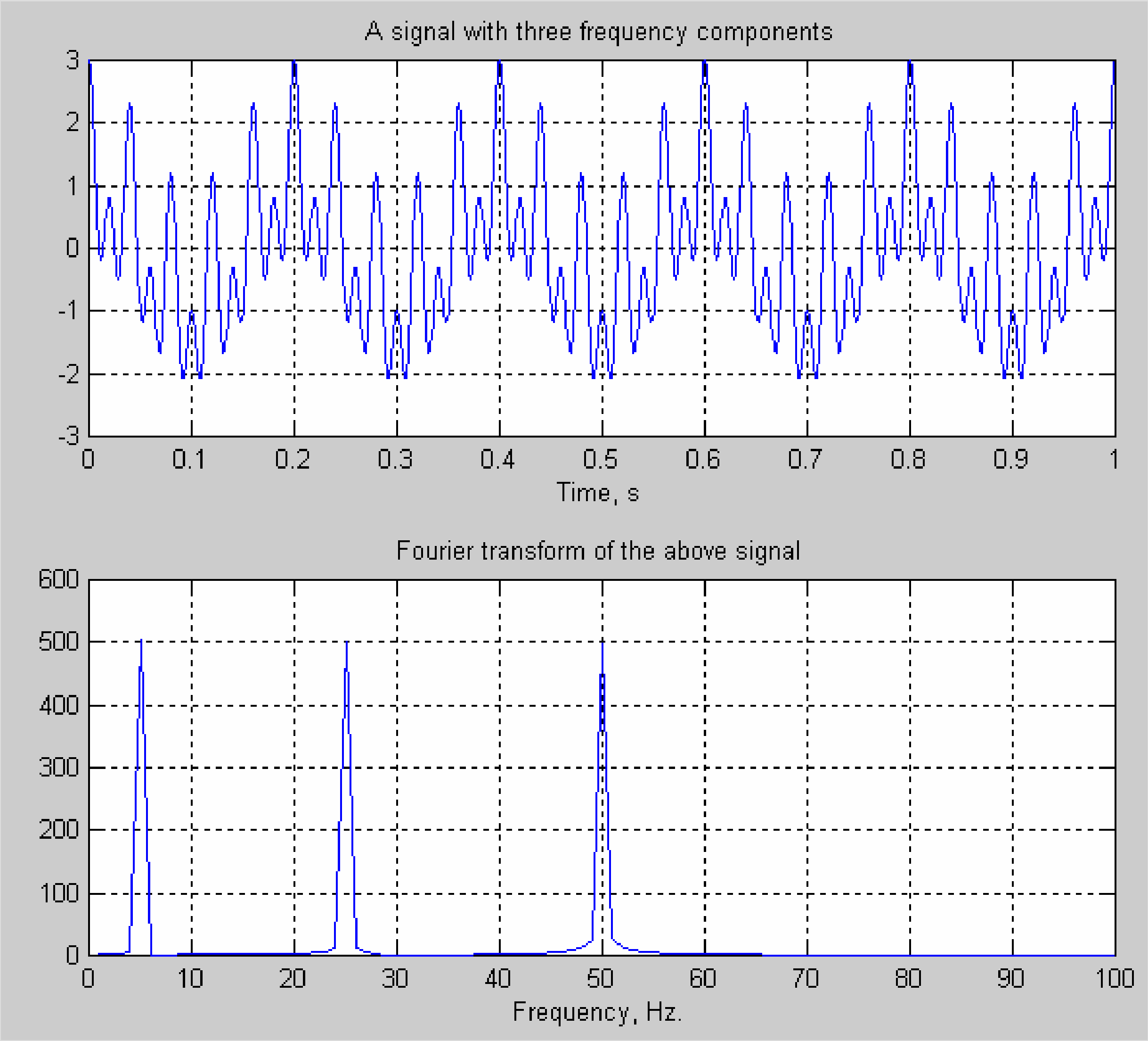
$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$$

$$x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

$$x_3(t) \overset{\mathcal{F}}{\longleftrightarrow} X_3(\omega)$$



$$x_4(t) = \cos(2\pi \cdot 5 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t)$$



$$x_4(t) \overset{\mathcal{F}}{\longleftrightarrow} X_4(\omega)$$

Fourier Transformation properties

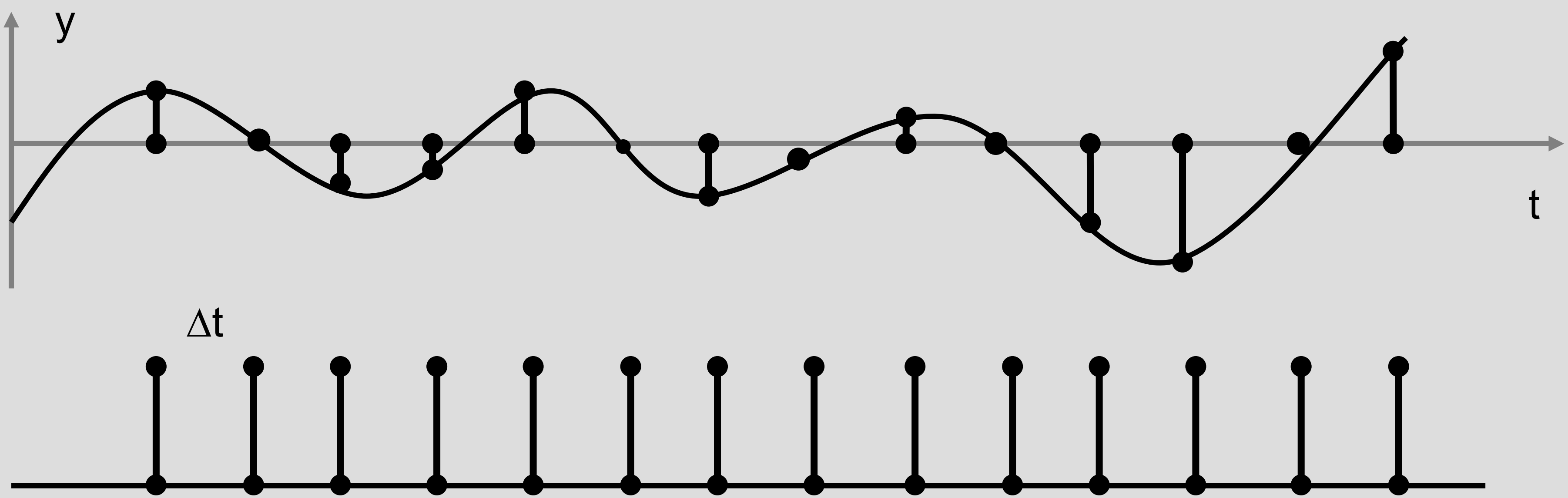
	Time	Frequency
Homogeneity	$a \cdot s(t)$	$a \cdot S(k)$
Additivity	$s(t) + u(t)$	$S(k)+U(k)$
Linearity	$a \cdot s(t) + b \cdot u(t)$	$a \cdot S(k)+b \cdot U(k)$
Time reversal	$s(-t)$	$S(-k)$
Multiplication *	$s(t) \cdot u(t)$	$\sum_{m=-\infty}^{\infty} S(k-m)U(m)$
Convolution *	$\frac{1}{T} \cdot \int_0^T s(t-\bar{t}) \cdot u(\bar{t}) d\bar{t}$	$S(k) \cdot U(k)$
Time shifting	$s(t-\bar{t})$	$e^{-j\frac{2\pi k \cdot \bar{t}}{T}} \cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi m t}{T}} \cdot s(t)$	$S(k - m)$

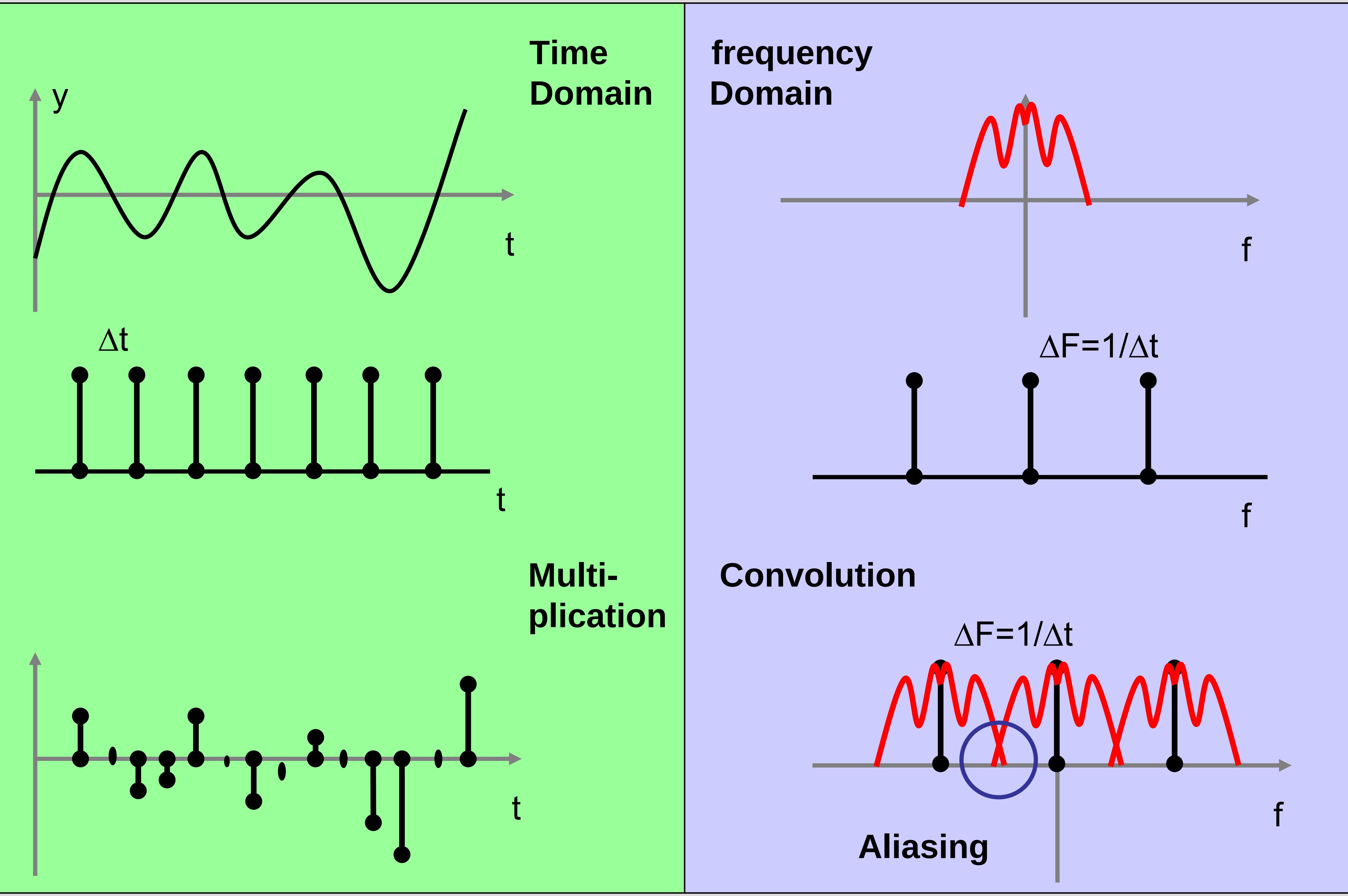
$$F(\omega) = \int f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int F(\omega) e^{j\omega t} d\omega$$

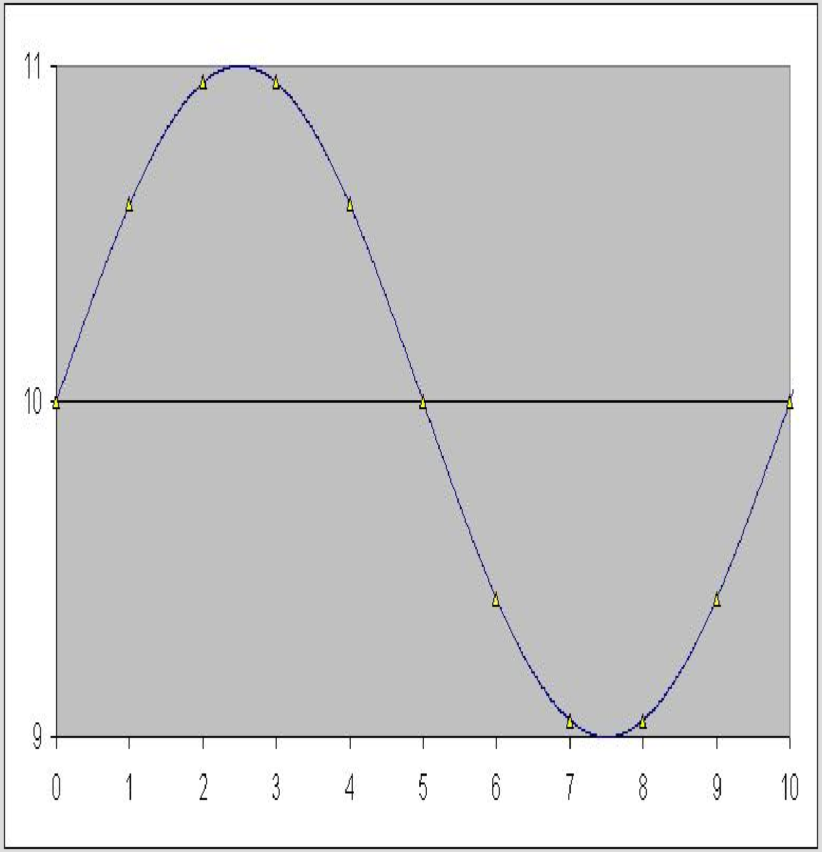
Multiplication in time domain becomes Convolution in Frequency domain

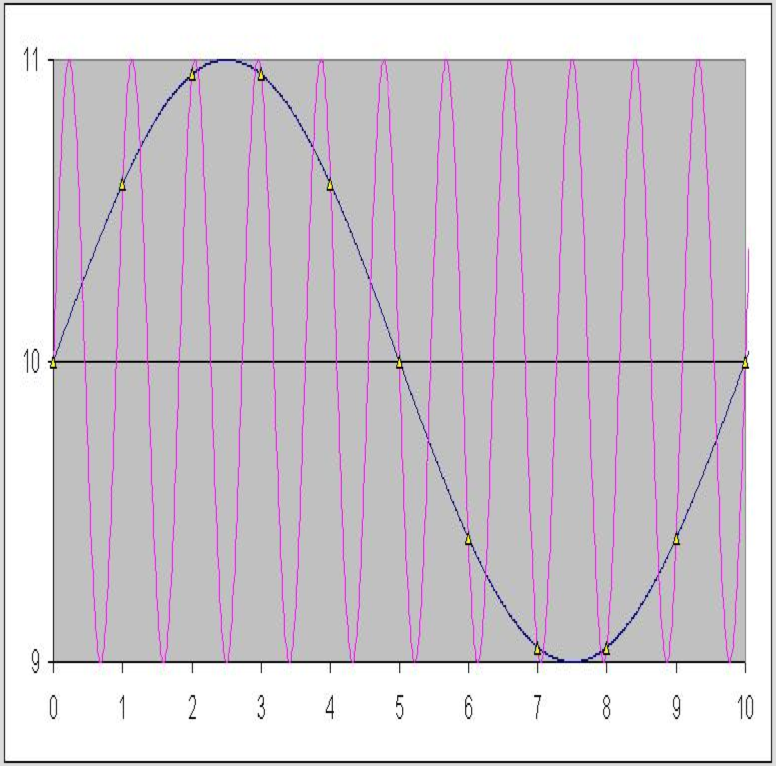
Properties of Sampling

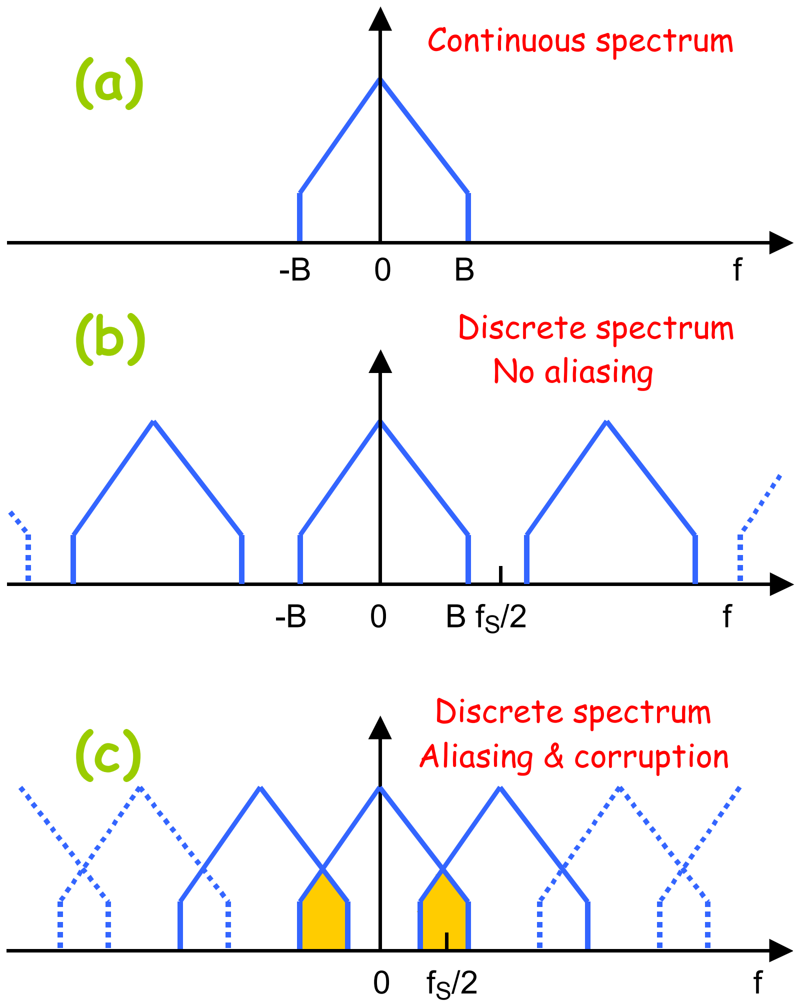



Time Domain
y
t
Δt
t
Multi-plication
t
frequency Domain
f
ΔF=1/Δt
f
Convolution
ΔF=1/Δt
f
Aliasing

Aliasing







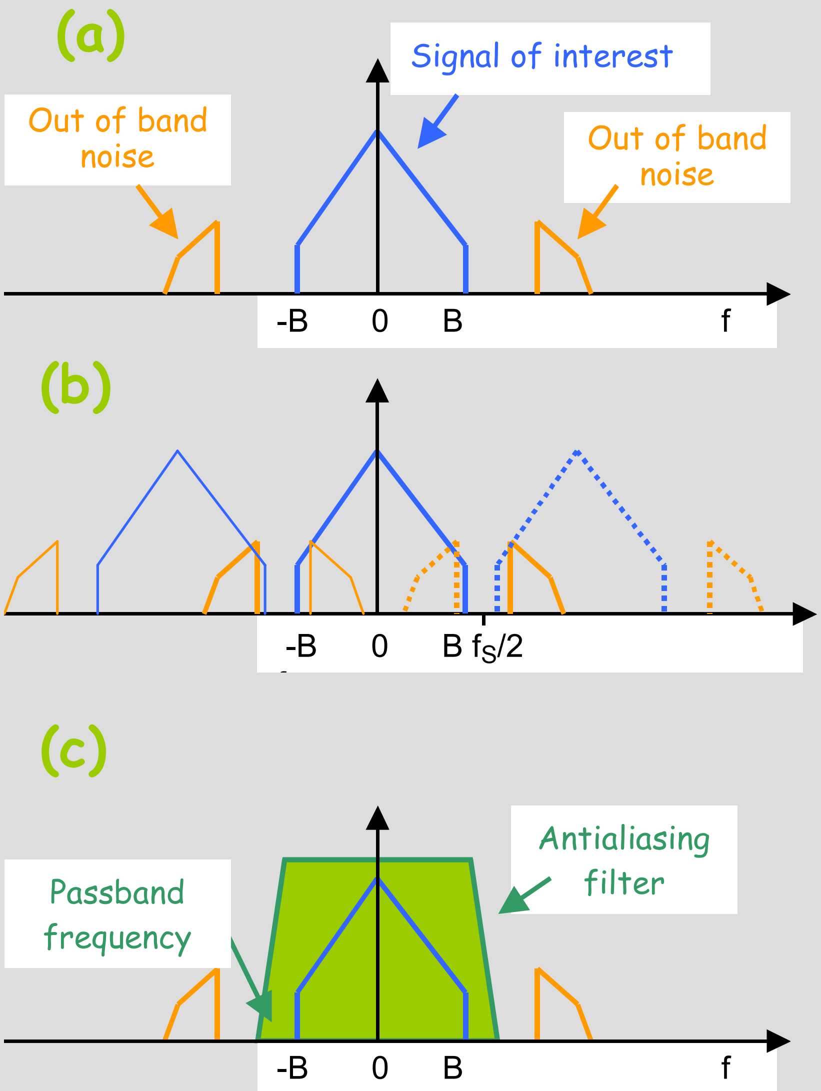
(a) Band-limited signal:
frequencies in [-B, B] (f_{MAX} = B).

(b) Time sampling ➡ frequency repetition.

$f_s > 2B$ ➡ no aliasing.

(c) $f_s \leqslant 2B$ ➡ aliasing !

Aliasing: signal ambiguity in frequency domain



(a),(b) *Out-of-band* noise can aliase into band of interest. Filter it before!

(c) Antialiasing filter

Passband: depends on bandwidth of interest.

Attenuation A_{MIN} : depends on

- ADC resolution (number of bits N).

$$A_{MIN,\ dB} \sim 6.02\ N + 1.76$$

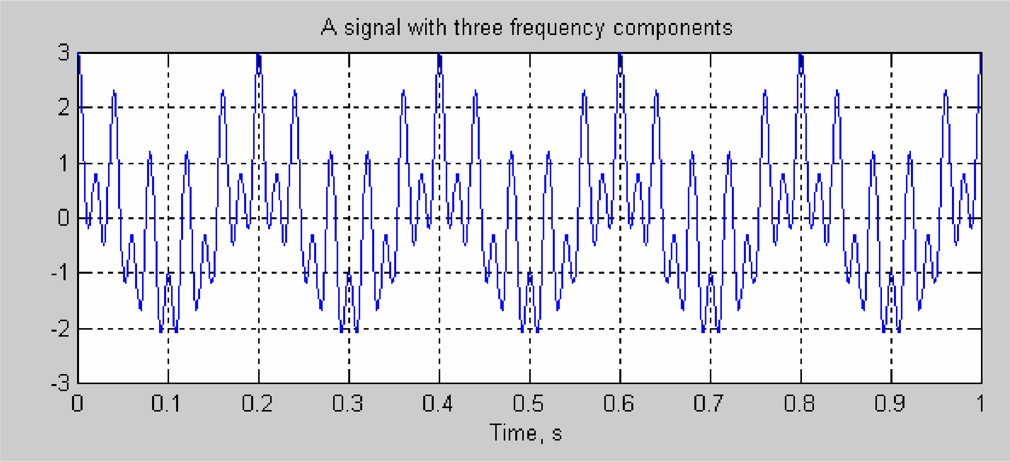
- Out-of-band noise magnitude.

Other parameters: ripple, stopband frequency...

- **FT identifies all spectral components present in the signal, however it does not provide any information regarding the temporal (time) localization of these components. Why?**

- Stationary signals' spectral characteristics do not change with time

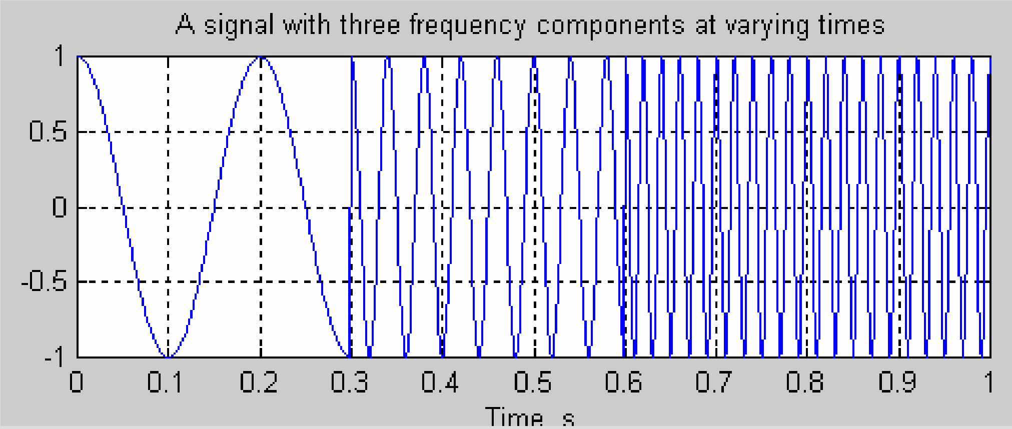
$$x_4(t) = \cos(2\pi \cdot 5 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t)$$



- Non-stationary signals have time varying spectra

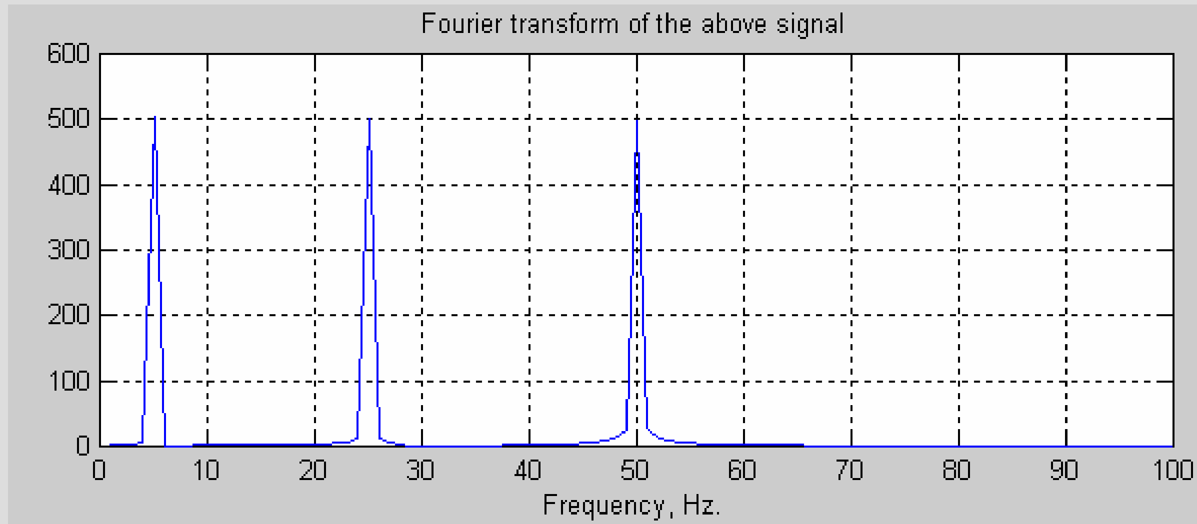
$$x_5(t) = [x_1 \oplus x_2 \oplus x_3]$$

$\oplus$ Concatenation

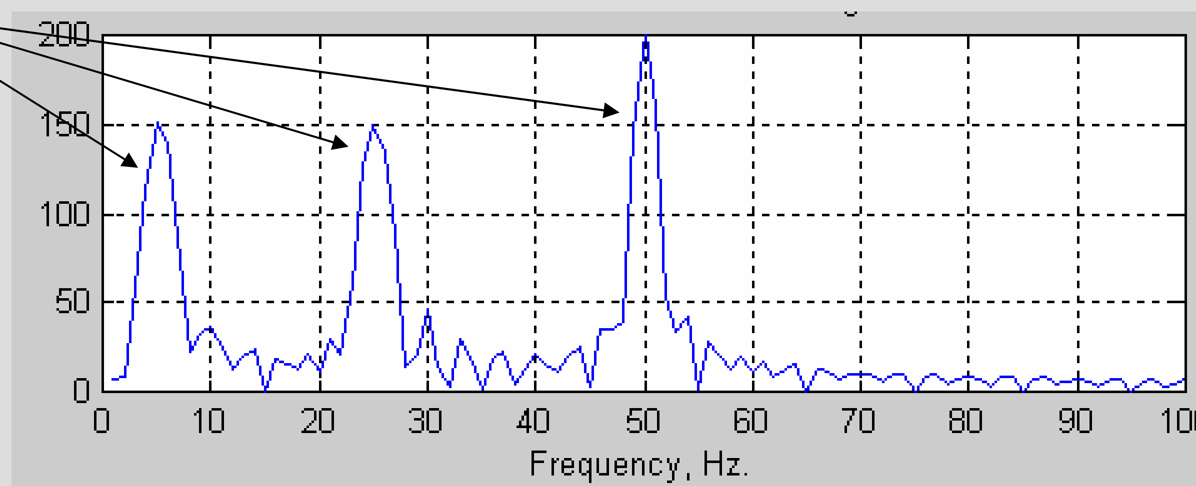


Perfect knowledge of what frequencies exist, but no information about where these frequencies are located in time

$X_4(\omega)$



$X_5(\omega)$

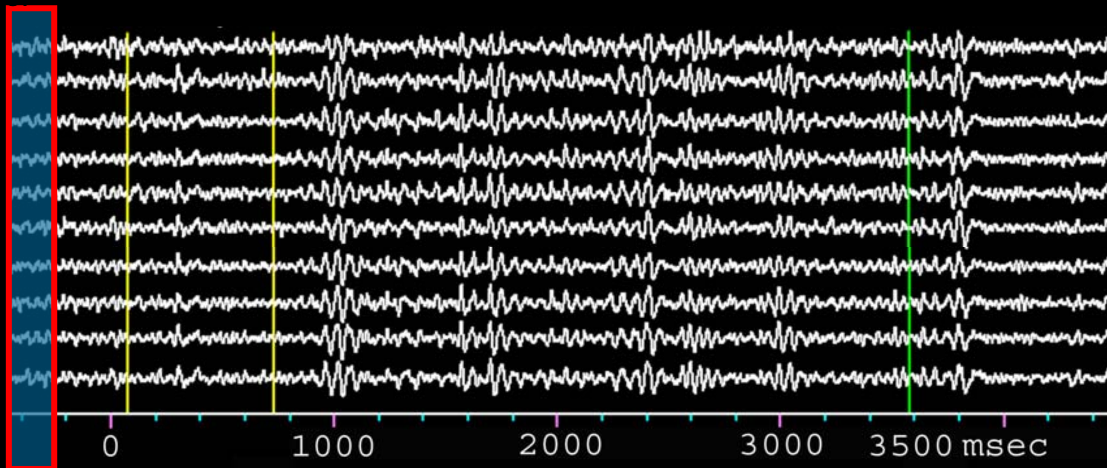


Short comings of the Fourier transformation

- Sinusoids and exponentials
  - Stretch into infinity in time → no time localization
  - Instantaneous in frequency → perfect spectral localization
  - ***Global*** analysis does not allow analysis of non-stationary signals

- Need a ***local*** analysis scheme for a time-frequency representation (TFR) of nonstationary signals
  - Windowed F.T. or Short Time F.T. (STFT) : Segmenting the signal into narrow time intervals, narrow enough to be considered stationary, and then take the Fourier transform of each segment, Gabor 1946.

Sliding Window Fourier transformation of the LFP signal

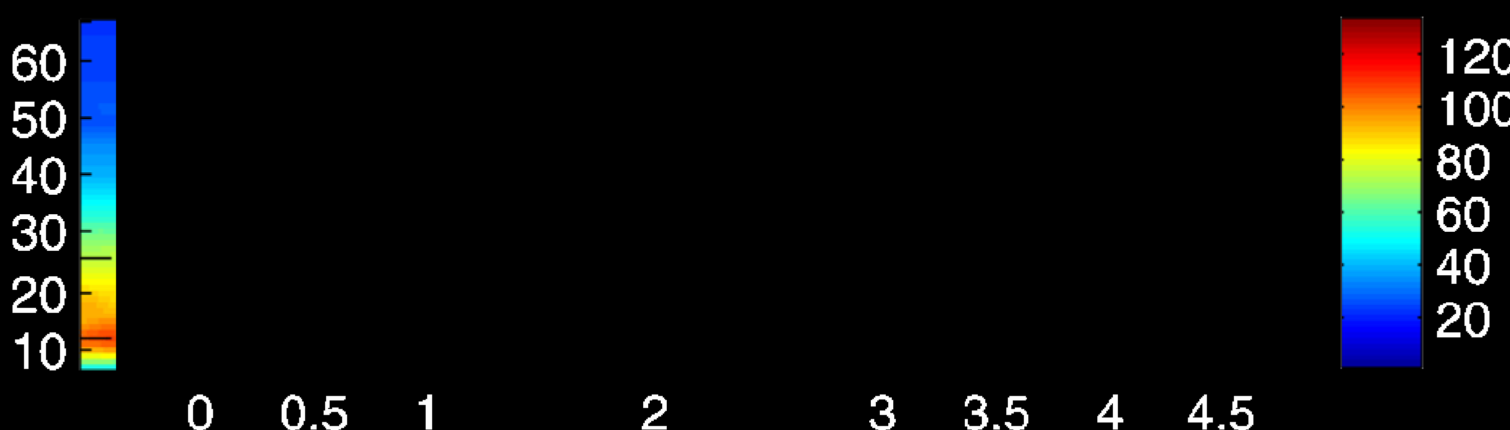


200ms

200ms

Fourier /Wavelet
Transformation

Mean Power - All Sessions Pooled

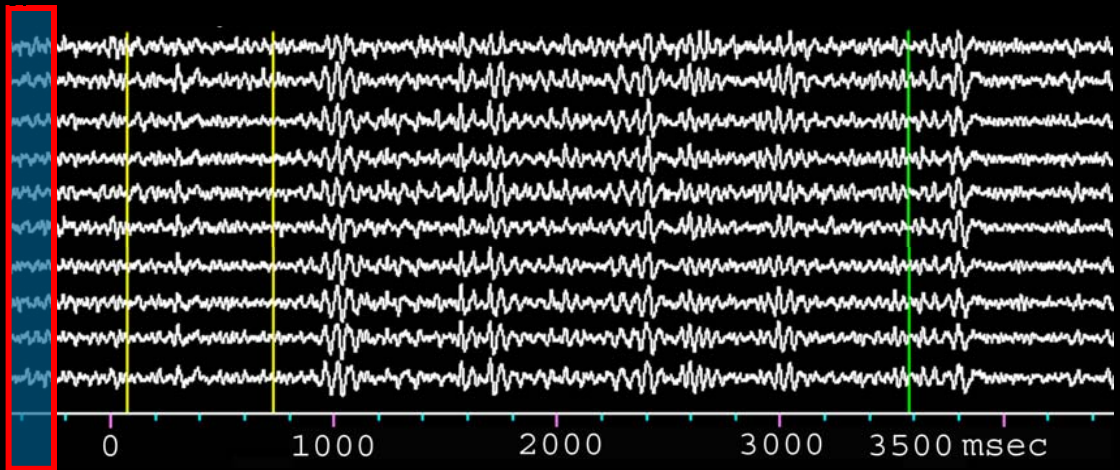


1. Choose a window function of finite length
2. Place the window on top of the signal at t=0
3. Truncate the signal using this window
4. Compute the FT of the truncated signal, save.
5. Incrementally slide the window to the right
6. Go to step 3, until window reaches the end of the signal

For each time location where the window is centered, we obtain a different FT

→ Hence, each FT provides the spectral information of a separate time-slice of the signal, providing simultaneous time and frequency information

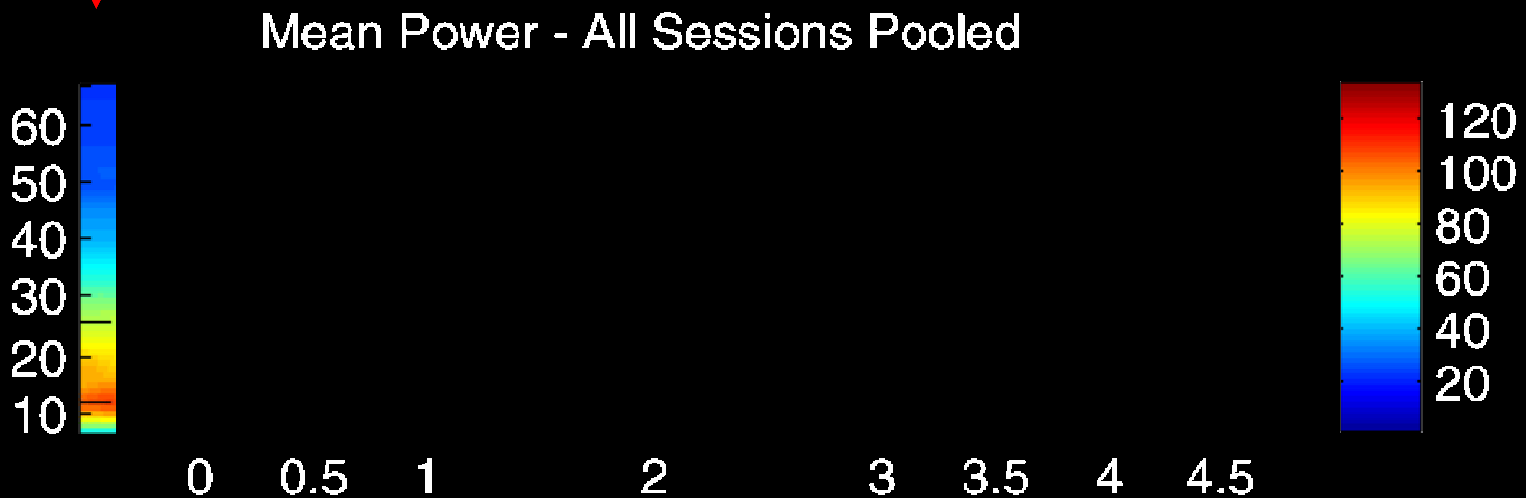
Sliding Window Fourier transformation of the LFP signal



200ms

200ms

Fourier /Wavelet Transformation



STFT

Time parameter

Frequency parameter

Signal to be analyzed

FT Kernel (basis function)

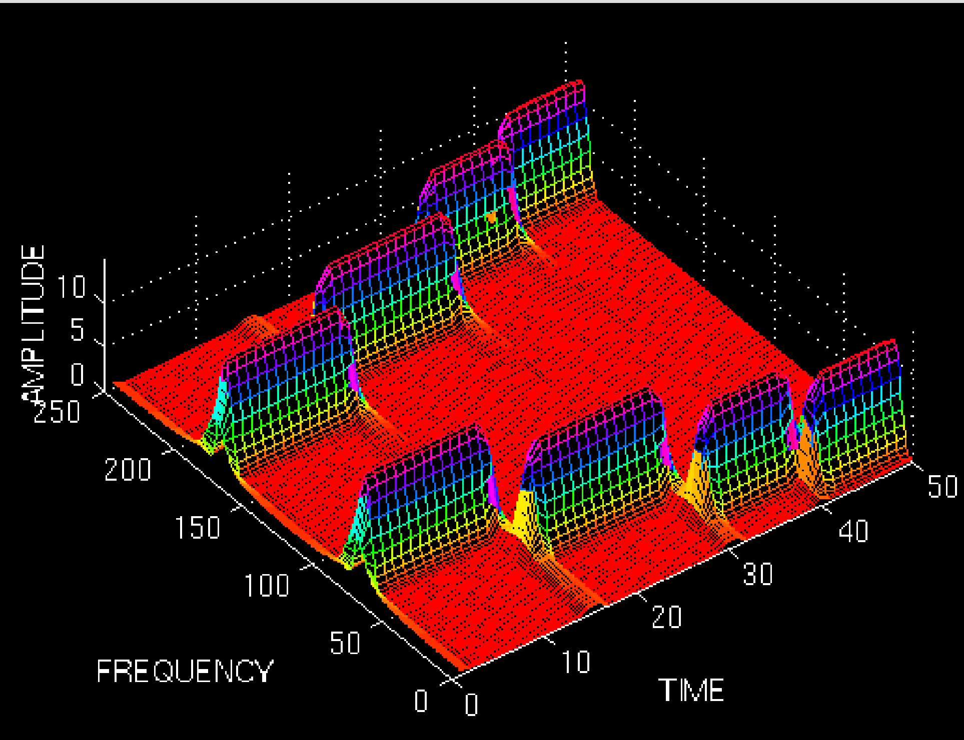
$$STFT_x^{\omega}(t',\omega)=\int_t \left[x(t)\cdot W(t-t')\right]\cdot e^{-j\omega t}\,dt$$

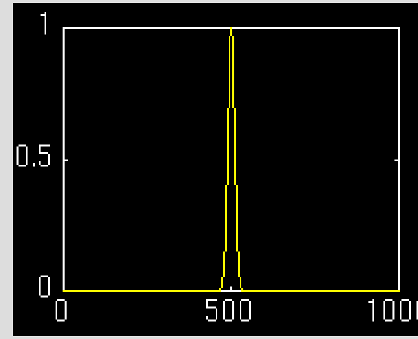
STFT of signal x(t): Computed for each window centered at t=t’

Windowing function

Windowing function centered at t=t’

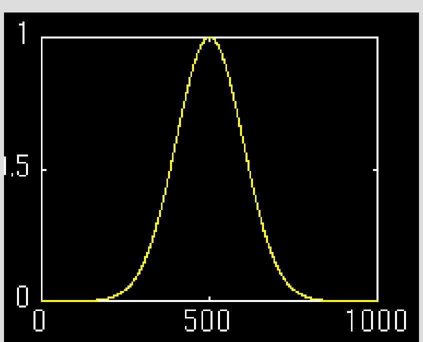
STFT At Work

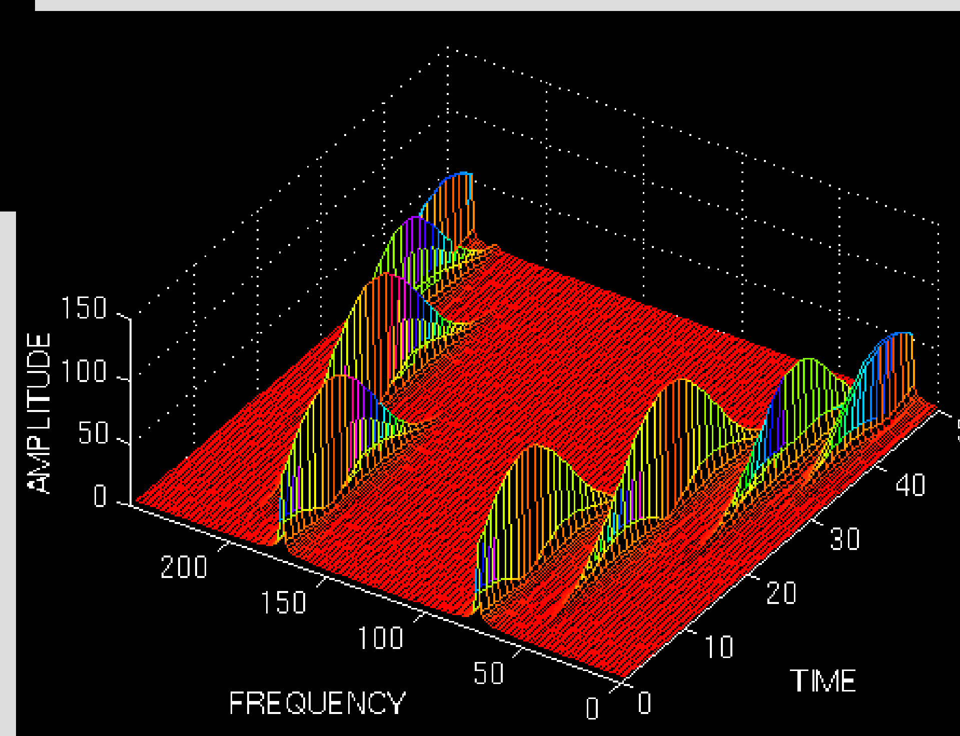




Sharp window in time

broader window in time





- STFT provides the time information by computing a different FTs for consecutive time intervals, and then putting them together
  - Time-Frequency Representation (TFR)
  - Maps 1-D time domain signals to 2-D time-frequency signals

- Consecutive time intervals of the signal are obtained by truncating the signal using a sliding windowing function

- How to choose the windowing function?
  - Wider window require less time steps → low time resolution
  - Also, window should be narrow enough to make sure that the portion of the signal falling within the window is stationary

$$STFT_x^{\omega}(t',\omega)=\int_t \left[x(t)\cdot W(t-t')\right]\cdot e^{-j\omega t}dt$$

Two extreme cases:

- *W(t)* infinitely long: $W(t)=1$

➔ STFT turns into FT, providing excellent frequency information (good frequency resolution), but no time information

$$\Delta t \cdot \Delta f \geq \frac{1}{4\pi}$$

Time resolution:

How well two spikes in time can be separated from each other in the transform domain

Frequency resolution:

How well two spectral components can be separated from each other in the transform domain

Both time and frequency resolutions cannot be arbitrarily high!!!

➔ We cannot precisely know at what time instance a frequency component is located. We can only know what *interval of frequencies* are present in which *time intervals*

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- Overcomes the preset resolution problem of the STFT by using a variable length window
- Analysis windows of different lengths are used for different frequencies:
  - Analysis of high frequencies ➔ Use narrower windows for better time resolution
  - Analysis of low frequencies ➔ Use wider windows for better frequency resolution
- This works well, if the signal to be analyzed mainly consists of slowly varying characteristics with occasional short high frequency bursts.
- Heisenberg principle still holds!!!
- The function used to window the signal is called ***the wavelet***

Translation parameter, measure of time

Scale parameter, measure of frequency

A normalization constant

Signal to be analyzed

$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int_t x(t)\psi^*\left(\frac{t-\tau}{s}\right)dt$$

Continuous wavelet transform of the signal *x(t)* using the analysis wavelet ψ(.)

The mother wavelet. All kernels are obtained by translating (shifting) and/or scaling the mother wavelet

Scale = 1/frequency

$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int_t x(t) \psi^* \left(\frac{t-\tau}{s} \right) dt$$

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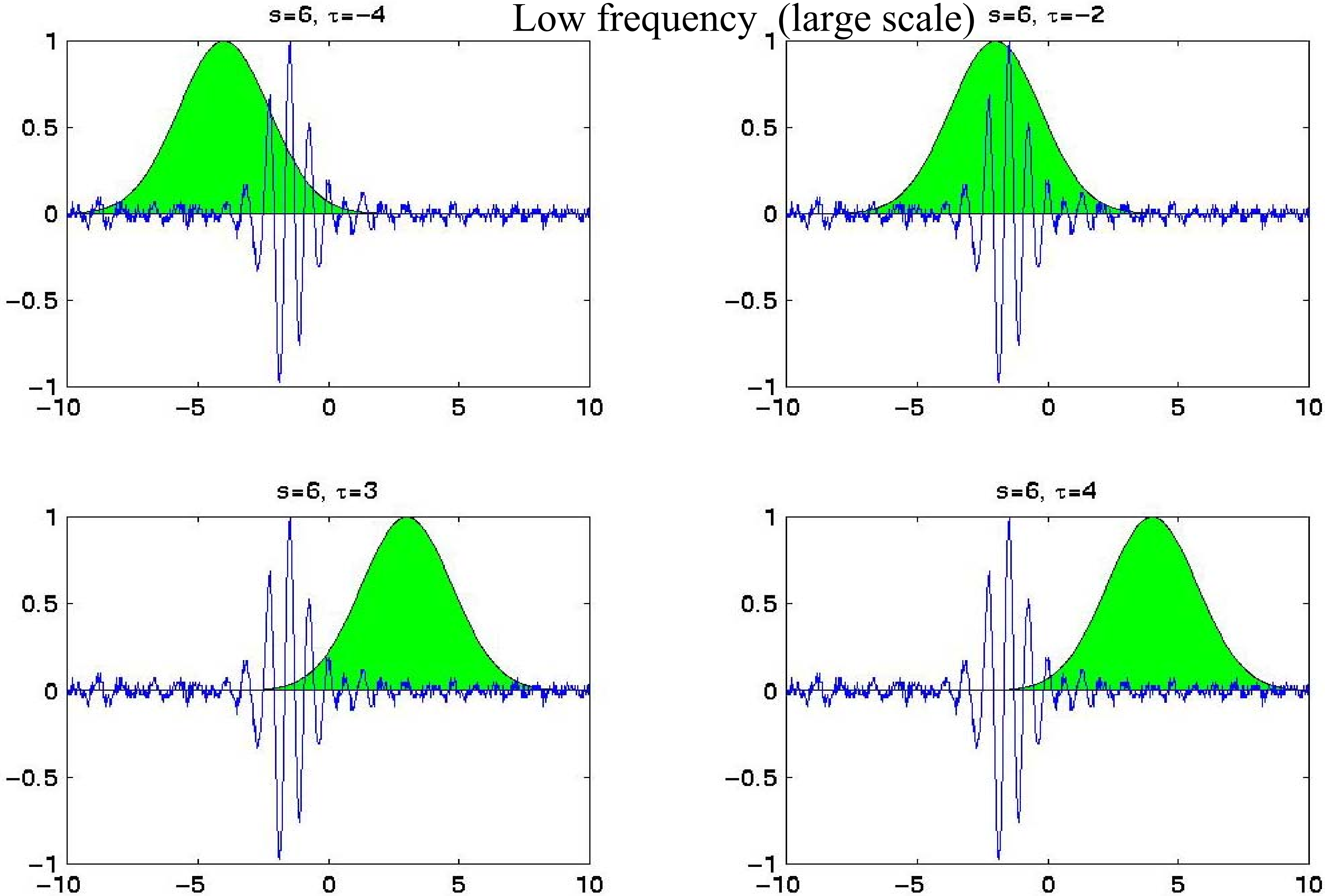
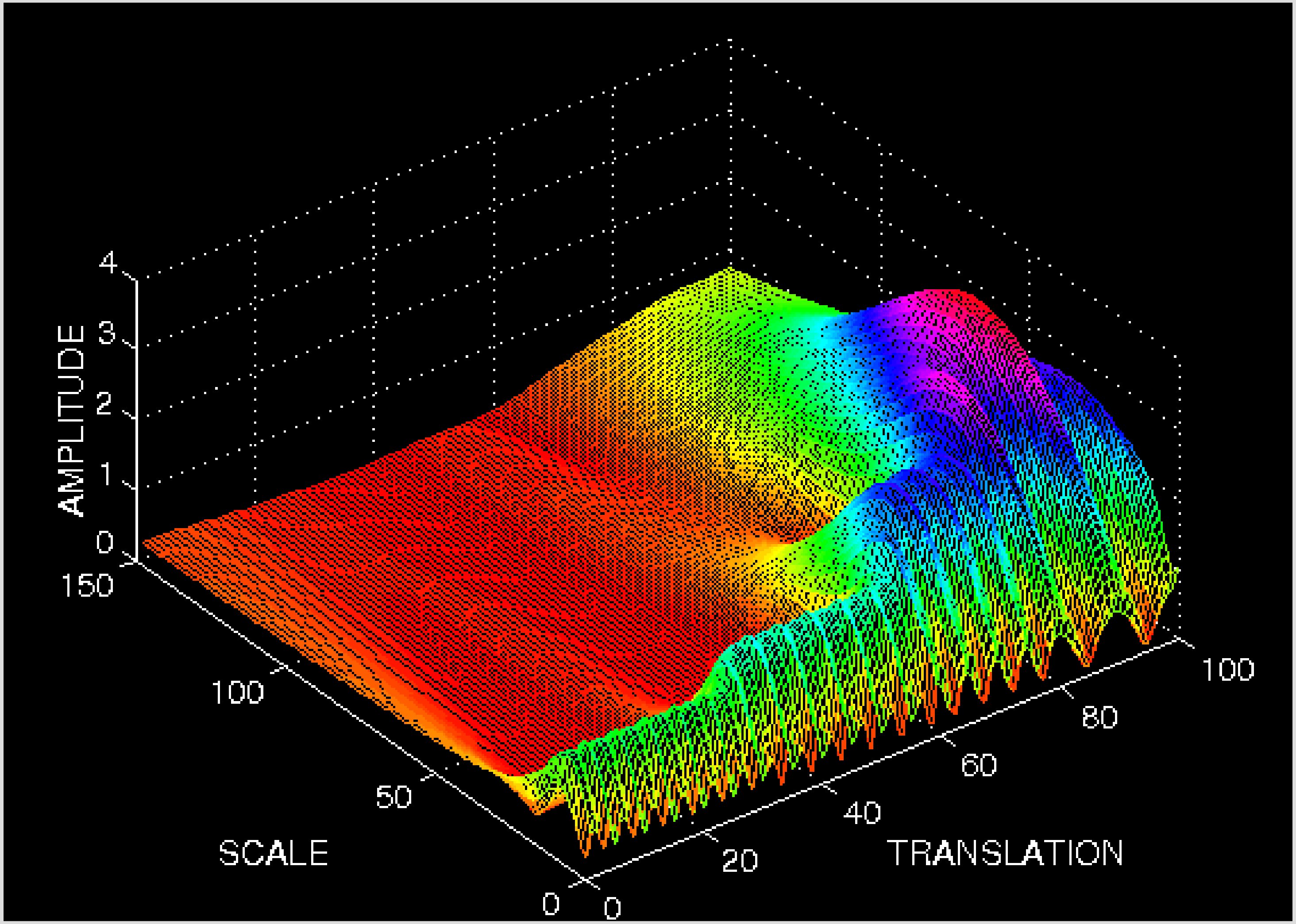
Low frequency (large scale)




30 Hz
20 Hz
10 Hz
5 Hz
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
200
400
600
800
1000
Time, ms

WT at Work



AMPLITUDE
4
3
2
1
0
150
100
50
0
SCALE
0
20
40
60
80
100
TRANSLATION

$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \bullet \psi^* \left(\frac{t - \tau}{s} \right) dt$$

Step 1: The wavelet is placed at the beginning of the signal, and set s=1 (the most compressed wavelet);

Step 2: The wavelet function at scale "1" is multiplied by the signal, and integrated over all times; then multiplied by $1/\sqrt{|s|}$

Step 3: Shift the wavelet to $t= \tau$, and get the transform value at $t= \tau$ and s=1;

Step 4: Repeat the procedure until the wavelet reaches the end of the signal;

Step 5: Scale s is increased by a sufficiently small value, the above procedure is repeated for all *s;*

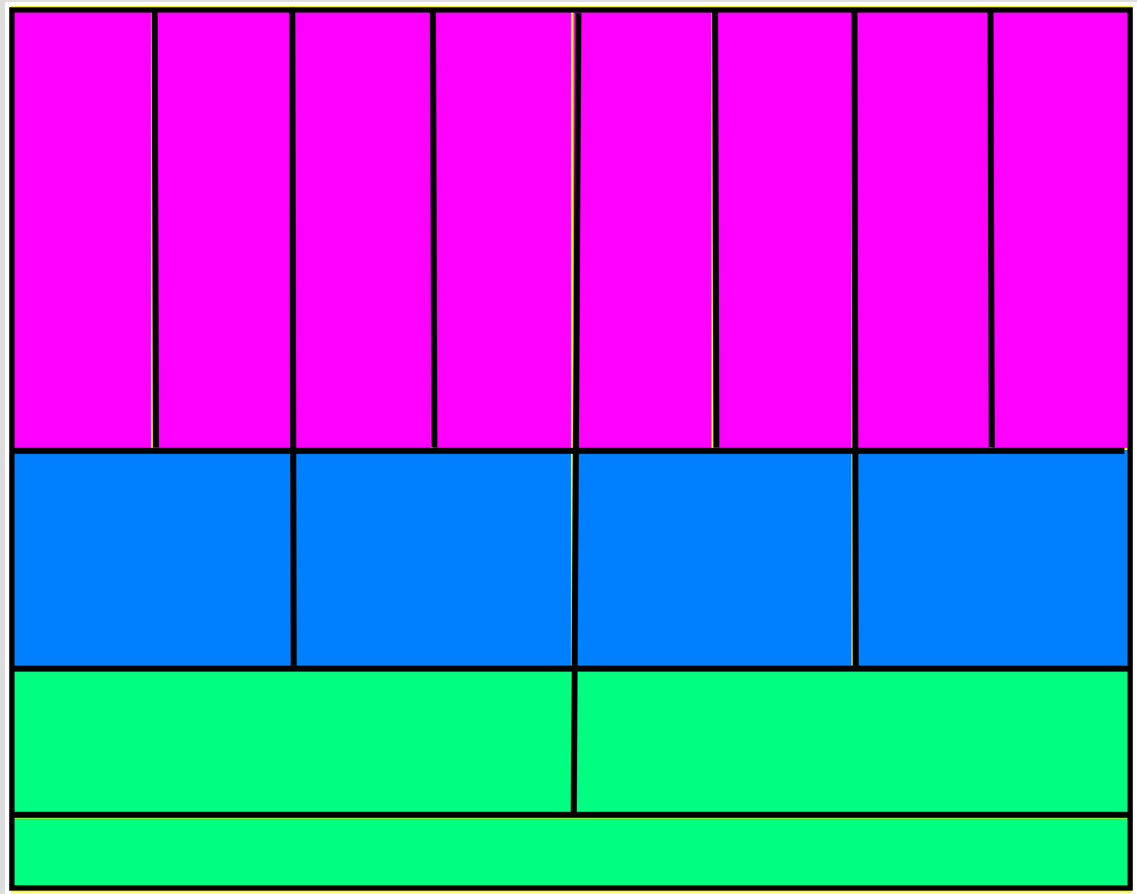
Step 6: Each computation for a given *s* fills the single row of the time-scale plane;

Step 7: CWT is obtained if all s are calculated.

RESOLUTION OF TIME & FREQUENCY

Better time resolution; Poor frequency resolution

Frequency

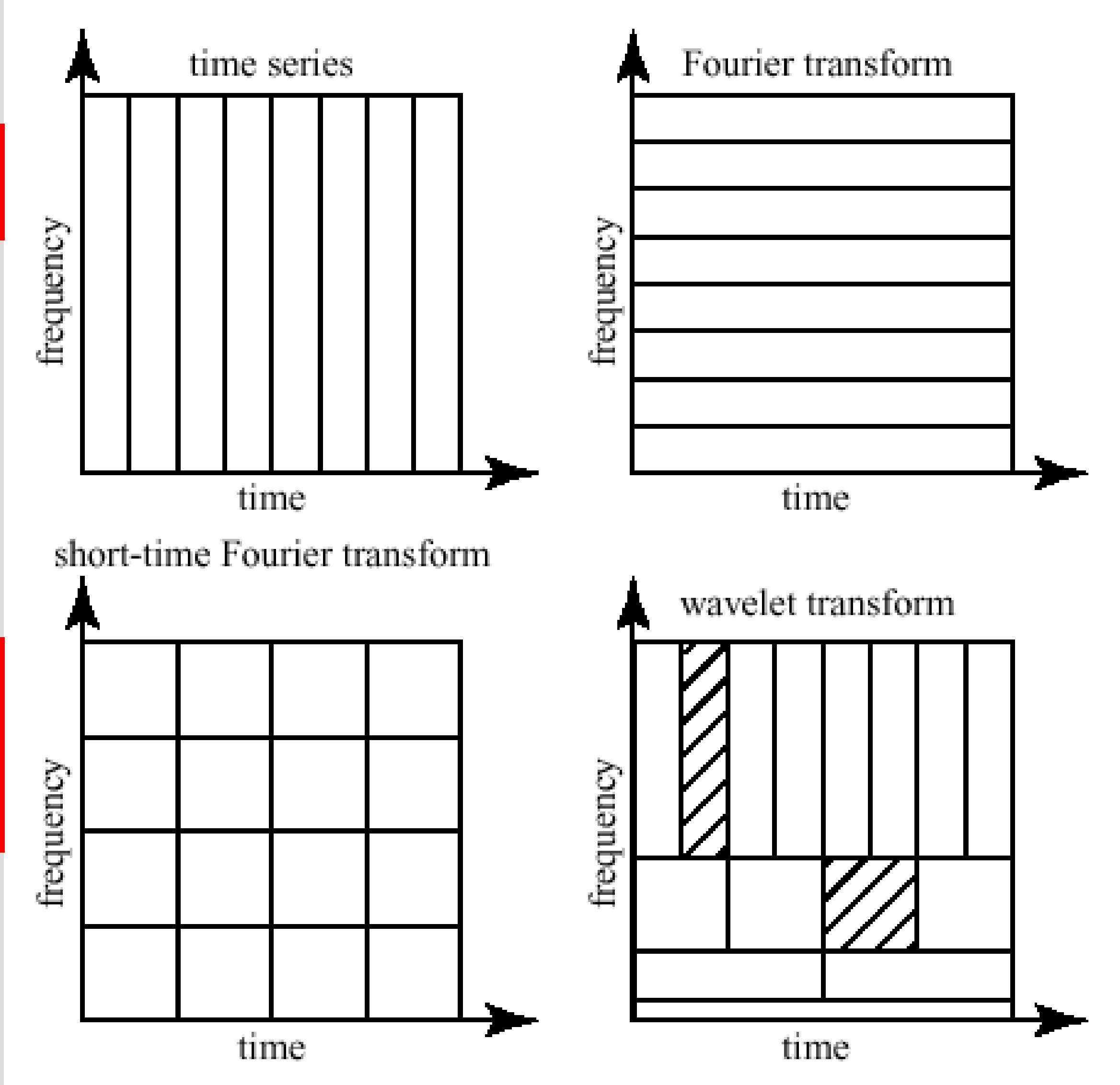


Time

Better frequency resolution; Poor time resolution

Regardless of the dimensions of the boxes, the areas of all boxes, both in STFT and WT, are the same and determined by **Heisenberg's inequality** .

Full time but no frequency info



Full frequency but no time info

frequency and time resolution always the same

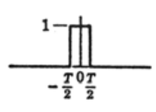
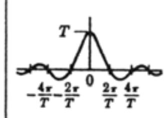
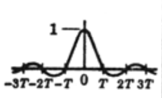
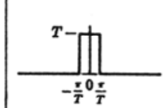
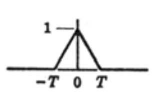
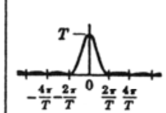
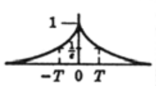
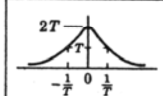
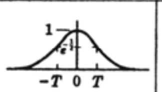
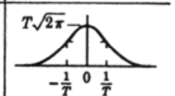
Higher time resolution for high frequencies

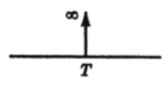
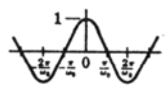
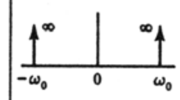
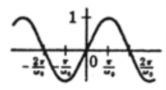
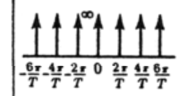
What have we learned

1. Concepts in time series Analysis
2. Basic Statistics
3. Auto Correlation
4. Auto Regressive Modeling
5. Fourier Transformation
6. Wavelet Transformation

The end

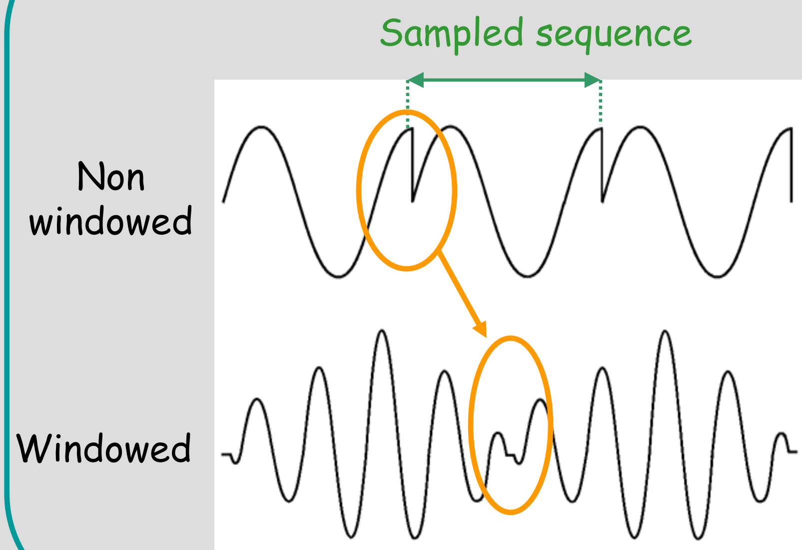
FT of main waveforms

	$f(t)=\int_{-\infty}^{\infty} F_F(i\omega)e^{i\omega t}\,\frac{d\omega}{2\pi}$	$F_F(i\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$	
	$\text{rect}\,\frac{t}{T}=\begin{cases}1 & (\lvert t\rvert<T/2)\\ 0 & (\lvert t\rvert>T/2)\end{cases}$	$T\,\text{sinc}\,\frac{\omega T}{2\pi}\equiv T\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}$	
	$\text{sinc}\,\frac{t}{T}\equiv\frac{\sin\frac{\pi t}{T}}{\frac{\pi t}{T}}$	$T\,\text{rect}\,\frac{\omega T}{2\pi}=\begin{cases}0 & (\lvert\omega\rvert<\frac{\pi}{T})\\ T & (\lvert\omega\rvert>\frac{\pi}{T})\end{cases}$	
	$\begin{cases}1-\frac{\lvert t\rvert}{T} & (\lvert t\rvert<T)\\ 0 & (\lvert t\rvert\geq T)\end{cases}$	$T\,\text{sinc}^2\frac{\omega T}{2\pi}\equiv T\left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^2$	
	$e^{-\frac{\lvert t\rvert}{T}}$	$\frac{2T}{(\omega T)^2+1}$	
	$e^{-\frac{1}{2}\left(\frac{t}{T}\right)^2}$	$\sqrt{2\pi}\,Te^{-\frac{1}{2}(\omega T)^2}$	

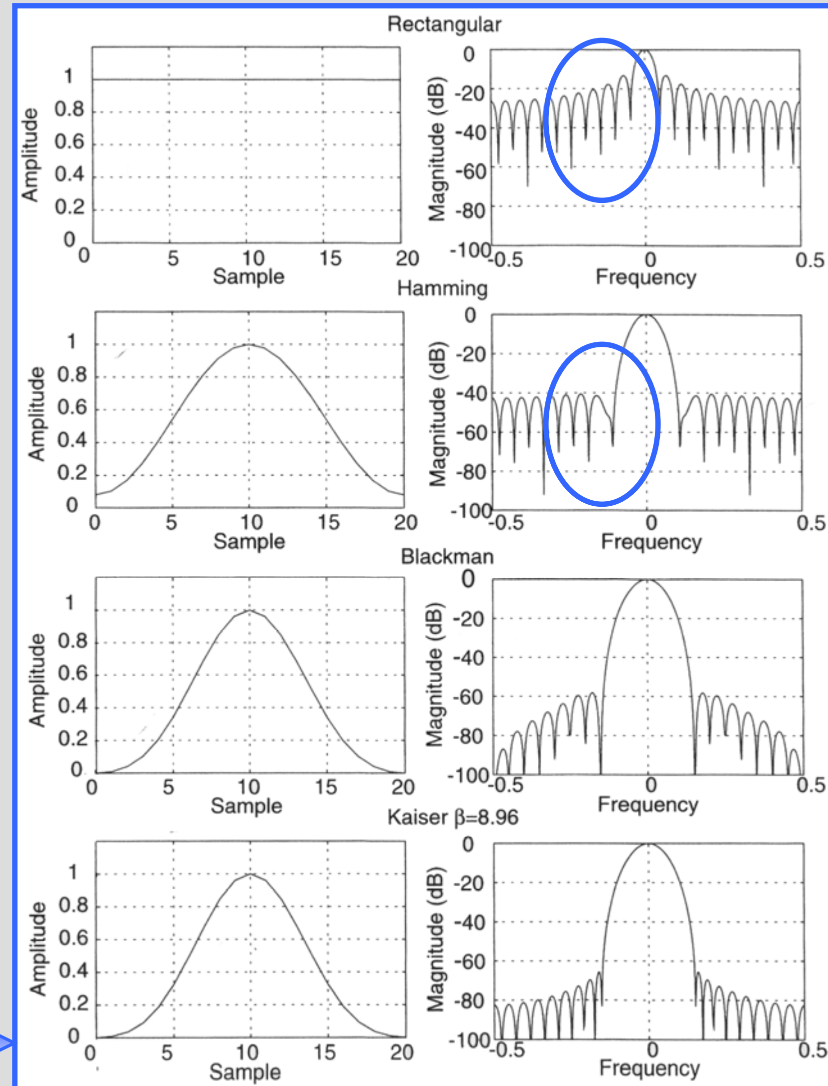
	$f(t)=\int_{-\infty}^{\infty} F_F(i\omega)e^{i\omega t}\,\frac{d\omega}{2\pi}$	$F_F(i\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$	
	$\delta(t-T)$	$e^{-i\omega t}$	(Complex)
	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
	$\sin\omega_0 t$	$\frac{\pi}{i}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	(Imaginary)
	$\sum_{k=-\infty}^{\infty}\delta(t-kT)\equiv\frac{1}{T}\sum_{j=-\infty}^{\infty}e^{2\pi ij\frac{t}{T}}$	$\frac{2\pi}{T}\sum_{j=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi j}{T}\right)=\sum_{k=-\infty}^{\infty}e^{ik\omega T}$	

Windowing reduces leakage by minimising sidelobes magnitude.

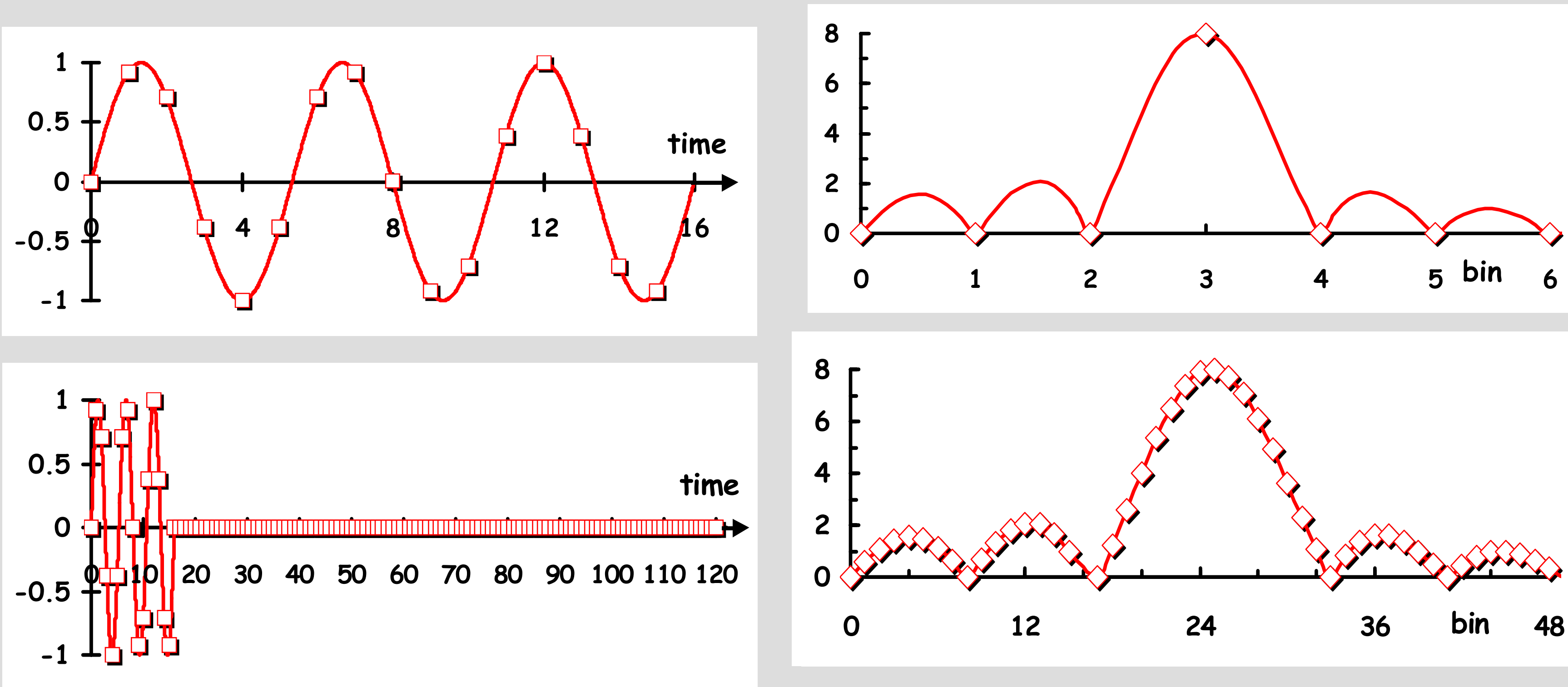
In time it reduces end-points discontinuities.



Some window functions



Improves DFT frequency inter-sampling spacing ("resolution").



After padding bins @ frequencies $f_m = \dfrac{m \cdot f_S}{N_S + L}$ N_S = original samples, L = padded.

DFT spectral resolution

Capability to distinguish two closely-spaced frequencies: not improved by zero-padding!.

Frequency inter-sampling spacing: increased by zero-padding (DFT "frequency span" unchanged due to same sampling frequency)

- Zero-padding in frequency domain increases sampling rate in time domain. Note: it works only if sampling theorem satisfied!
- Additional reason for zero-padding: to reach a power-of-two input samples number (see FFT).

NOTE

Apply zero-padding **after** windowing (if any)! Otherwise stuffed zeros will partially distort window function.

WAVELET FAMILY PROPERTIES

Property	morl	mexh	meyr	haar	*db*N	symN	coifN	biorNr.Nd	rbioNr.Nd	gaus	dmey	cgau	cmor	fbsp	shan
Crude	•	•								•		•	•	•	•
Infinitely regular	•	•	•							•		•	•	•	•
Arbitrary regularity					•	•	•	•	•						
Compactly supported orthogonal				•	•	•	•								
Compactly supported biothogonal								•	•						
Symmetry	•	•	•	•				•	•	•	•	•	•	•	•
Asymmetry					•										
Near symmetry						•	•								
Arbitrary number of vanishing moments					•	•	•	•	•						
Vanishing moments for ϕ							•								
Existence of ϕ			•	•	•	•	•	•	•						
Orthogonal analysis			•	•	•	•	•								
Biorthogonal analysis			•	•	•	•	•	•	•						
Exact reconstruction	≈	•	•	•	•	•	•	•	•	•	≈	•	•	•	•
FIR filters				•	•	•	•	•	•		•				
Continuous transform	•	•	•	•	•	•	•	•	•	•					
Discrete transform			•	•	•	•	•	•	•		•				
Fast algorithm				•	•	•	•	•	•		•				
Explicit expression	•	•		•				For splines	For splines	•		•	•	•	•
Complex valued												•	•	•	•
Complex continuous transform												•	•	•	•
FIR-based approximation											•				