

Theoretical Neuroscience & Complex Systems



05- 27. Aug. 2006
Frankfurt, Germany

Gordon Pipa

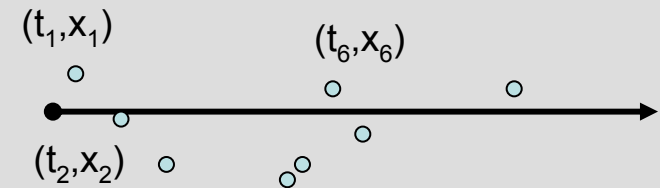
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1. Concepts in time series Analysis
2. Basic Statistics
3. Auto Correlation
4. Auto Regressive Modeling
5. Fourier Transformation
6. Wavelet Transformation

- A *time series* is a set of data pairs

$$(t_a, x_a), a = 1, 2, 3, \dots, N$$



- t is the time, x is the data value
- x can be anything, e.g. voltage, height, temperature ...

- Usually, times are assumed error-free \rightarrow Data = Signal + Error

$$x(t) = f(t) + \varepsilon$$

1. **Visual Inspection !!**

- Plot x as a function of t and Explore

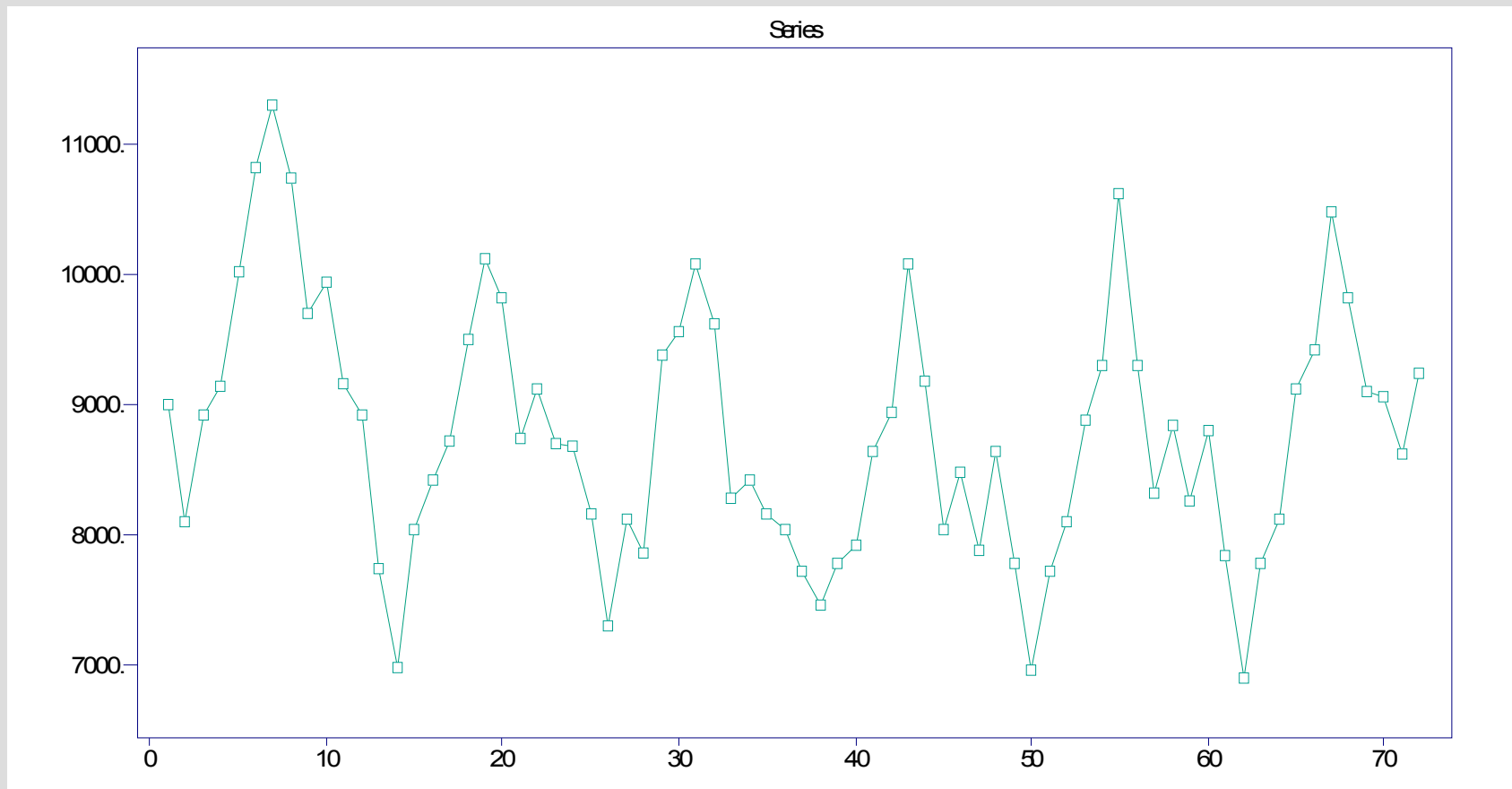
2. **Properties of interest**

- Basic Properties like mean and std
- Trends
- periodic components /limit cycles
- Auto correlation

3. **Explore data with sophisticated Analysis techniques**

Accidental Deaths in U.S.A. (DEATHS.TSM)

- Monthly totals: January 1973 to December 1978.
- Strong seasonal pattern: high in July, low in Feb.



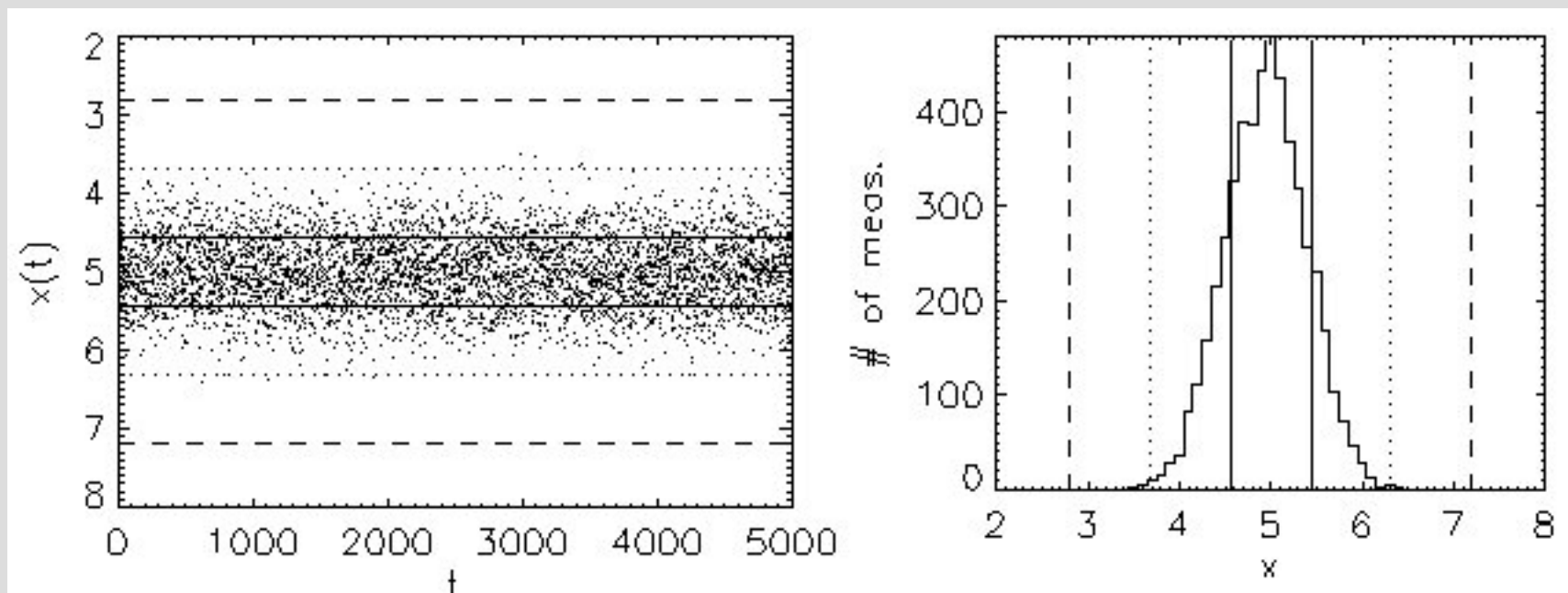
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Real underlying process

- μ : Mean = expected value
- σ : Standard deviation = expected variation from mean

Estimation

- $\hat{\mu}$: Average = estimated
- $\hat{\sigma}$: Sample standard deviation



Real underlying process

- μ : Mean = expected value
- σ : Standard deviation = expected variation from mean

Estimation

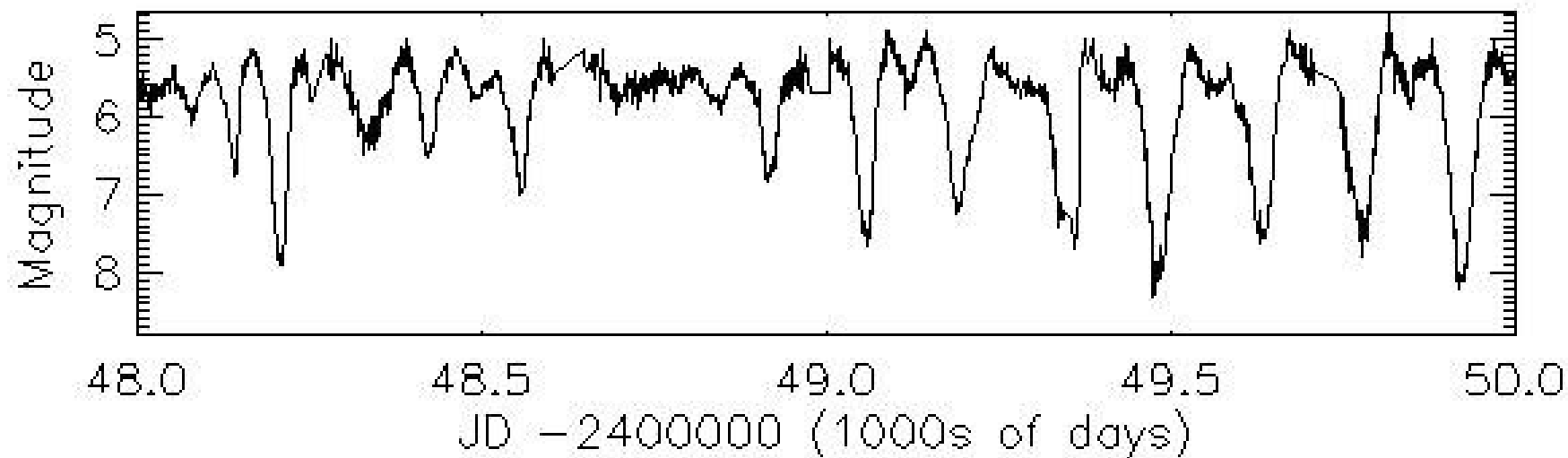
- $\hat{\mu}$: Average = estimated
- $\hat{\sigma}$: Sample standard deviation d

$$\bar{x} = \frac{1}{N} \sum x$$

$$s = \sqrt{\frac{1}{N-1} \sum (x - \bar{x})^2}$$

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- For a range of “periods” (τ), compare each data point $x(t)$ to a point $x(t+\tau)$
- The value of the correlation function at each τ is a function of the average difference between the points



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Stationary signals:

For a stationary signal differences in time are just dependent on a lag and independent of t itself:

Ergodic signals:

For an ergodic signal the population statistics is the same as statistics over time

- an ergodic signal must visit all potentially existing points in the phase space in an infinite amount of time
- Attractor states or limit cycle does not yield time series that are ergodic !!!

Auto Regressive Moving Average (ARMA) processes, are an important class of linear time series models. They provide a flexible parametric structure to approximate the behavior of stationary processes, and lead to a prediction theory that is relatively simple and elegant.

Without loss of generality, we assume $\{X_t\}$ has zero mean, since if $\{Y_t\}$ has mean μ , $X_t = Y_t - \mu$ has mean zero.

The AR Process

One of the most intuitive ways to model the behavior of a time series, is to regress X_t on its past p values, say. The resulting model is called an AutoRegression of order p , or AR(p):

$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + Z_t, \quad \{Z_t\} \sim \text{WN}(0, \sigma^2).$$

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December, 21, 1807



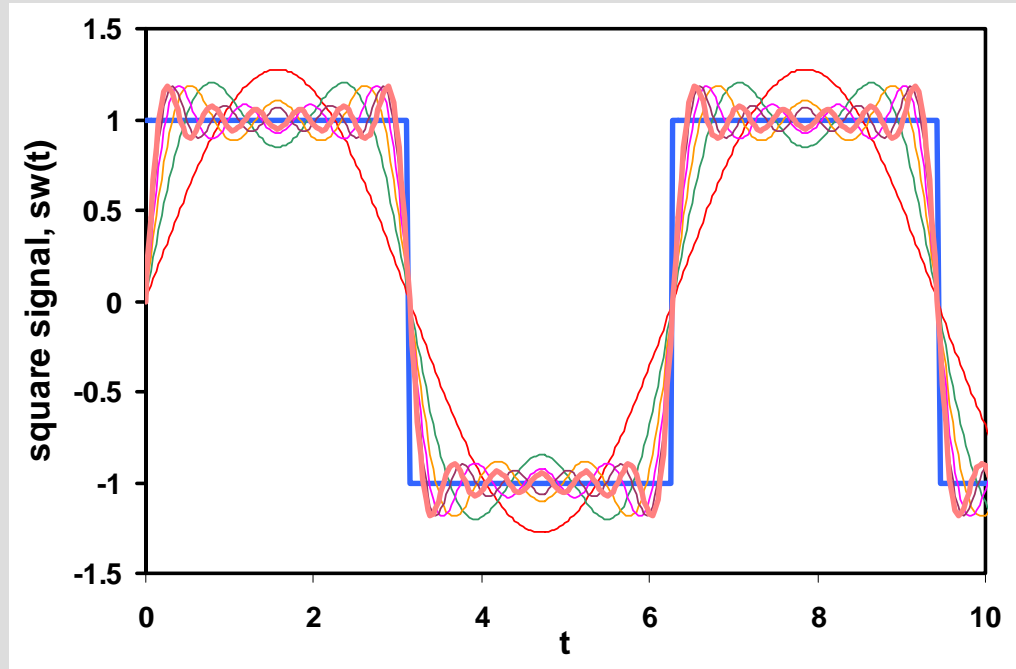
Jean B. Joseph Fourier
(1768-1830)

“An arbitrary function, continuous or with discontinuities, defined in a finite interval by an arbitrarily capricious graph can always be expressed as a sum of sinusoids”

J.B.J. Fourier

Square wave reconstruction from spectral terms

$$sw(t) = \sum_{k=1}^{\infty} \frac{1}{k} \sin(k\pi t)$$



Convergence may be slow ($\sim 1/k$) - ideally need infinite terms.

Overshoot exist at each discontinuity

$$sw_{79}(t) = \sum_{k=1}^{79} [-b_k \cdot \sin(kt)]$$



- Complex function representation through simple building blocks
 - Basis functions

$$\textit{Complex Function} = \sum_i (\textit{weight})_i \bullet (\textit{Simple Function})_i$$

- Using sinusoids as building blocks → Fourier transform
 - Frequency domain representation of the function

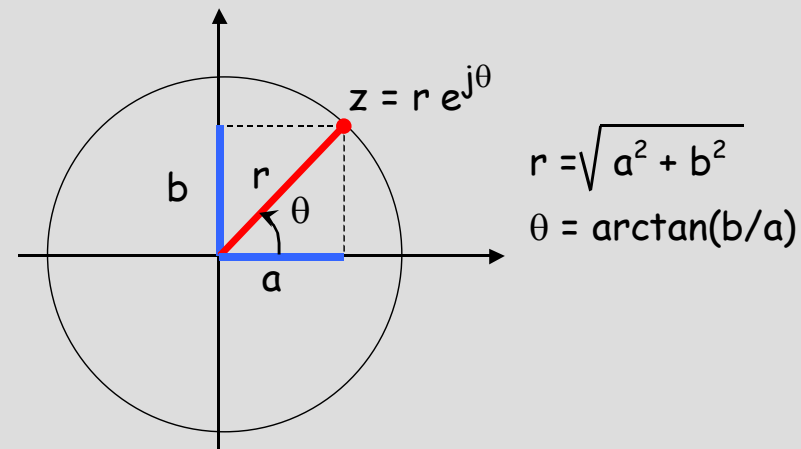
$$F(\omega) = \int f(t) e^{-j\omega t} dt \quad f(t) = \frac{1}{2\pi} \int F(\omega) e^{j\omega t} d\omega$$

- Recall that FT uses complex exponentials (sinusoids) as building blocks.

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

$$F(\omega) = \int f(t) e^{-j\omega t} dt \quad f(t) = \frac{1}{2\pi} \int F(\omega) e^{j\omega t} d\omega$$

- For each frequency of complex exponential, the sinusoid at that frequency is compared to the signal.
- If the signal consists of that frequency, the correlation is high
→ large FT coefficients.

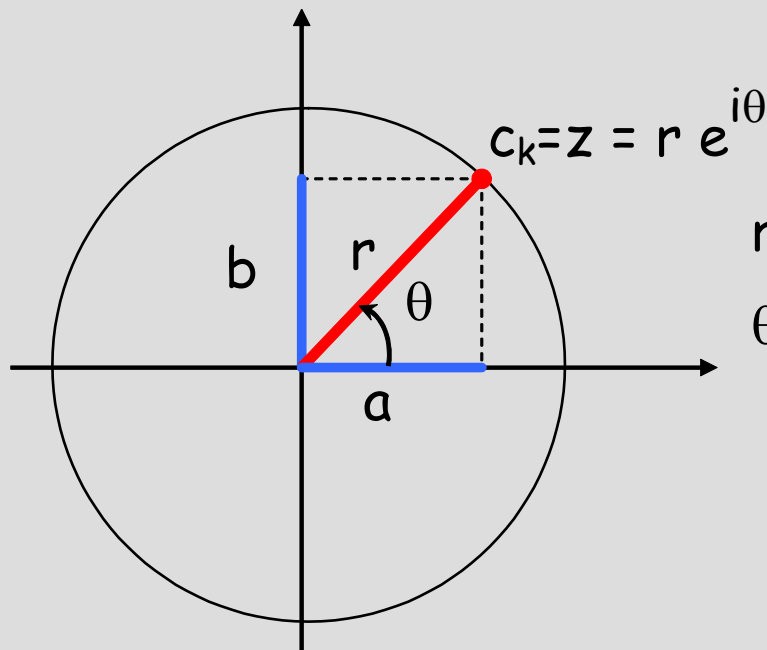


Euler's notation:

$$e^{-it} = (e^{it})^* = \cos(t) - i \cdot \sin(t)$$

$$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-ik\omega t} dt$$

$$c_k = r (\cos(\omega t) - i \sin(\omega t))$$

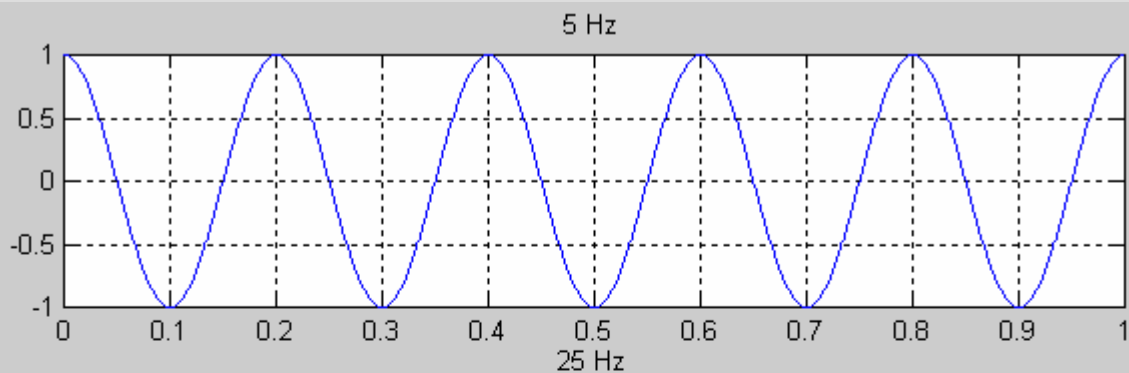


$$r = \sqrt{a^2 + b^2}$$

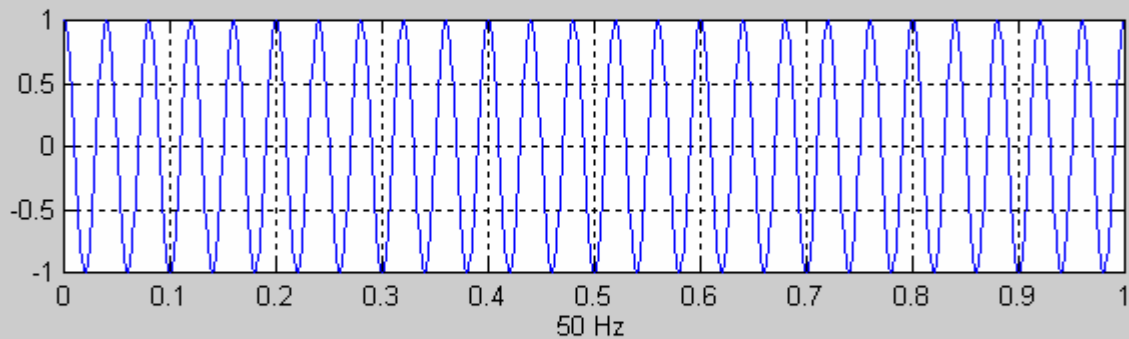
$$\theta = \arctan(b/a)$$

IF $s(t)$ is real \rightarrow Note: $c_{-k} = (c_k)^*$

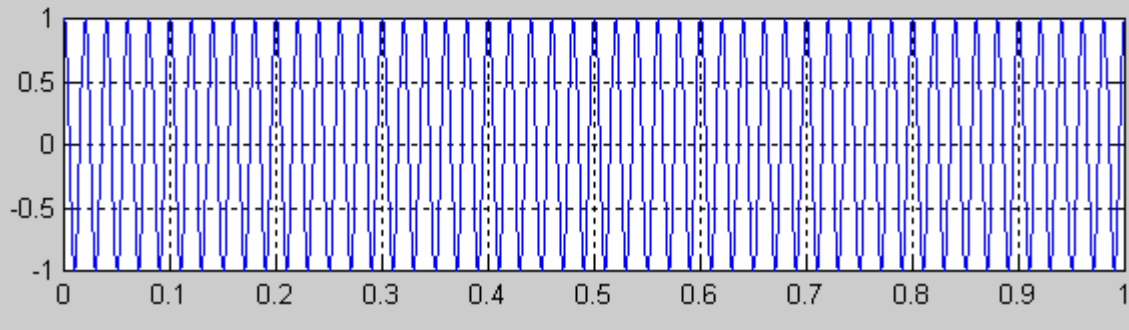
$$x_1(t) = \cos(2\pi \cdot 5 \cdot t)$$



$$x_2(t) = \cos(2\pi \cdot 25 \cdot t)$$



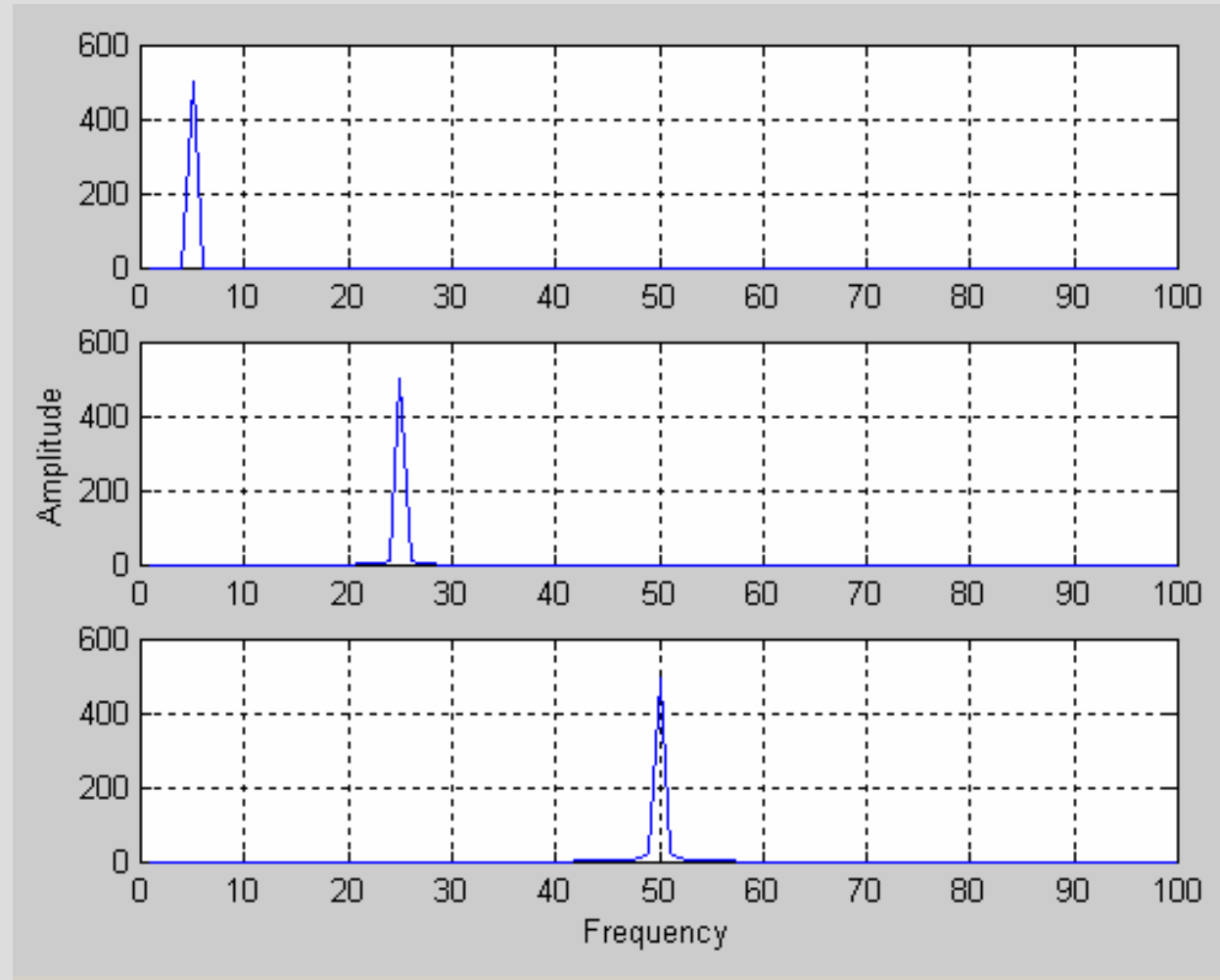
$$x_3(t) = \cos(2\pi \cdot 50 \cdot t)$$



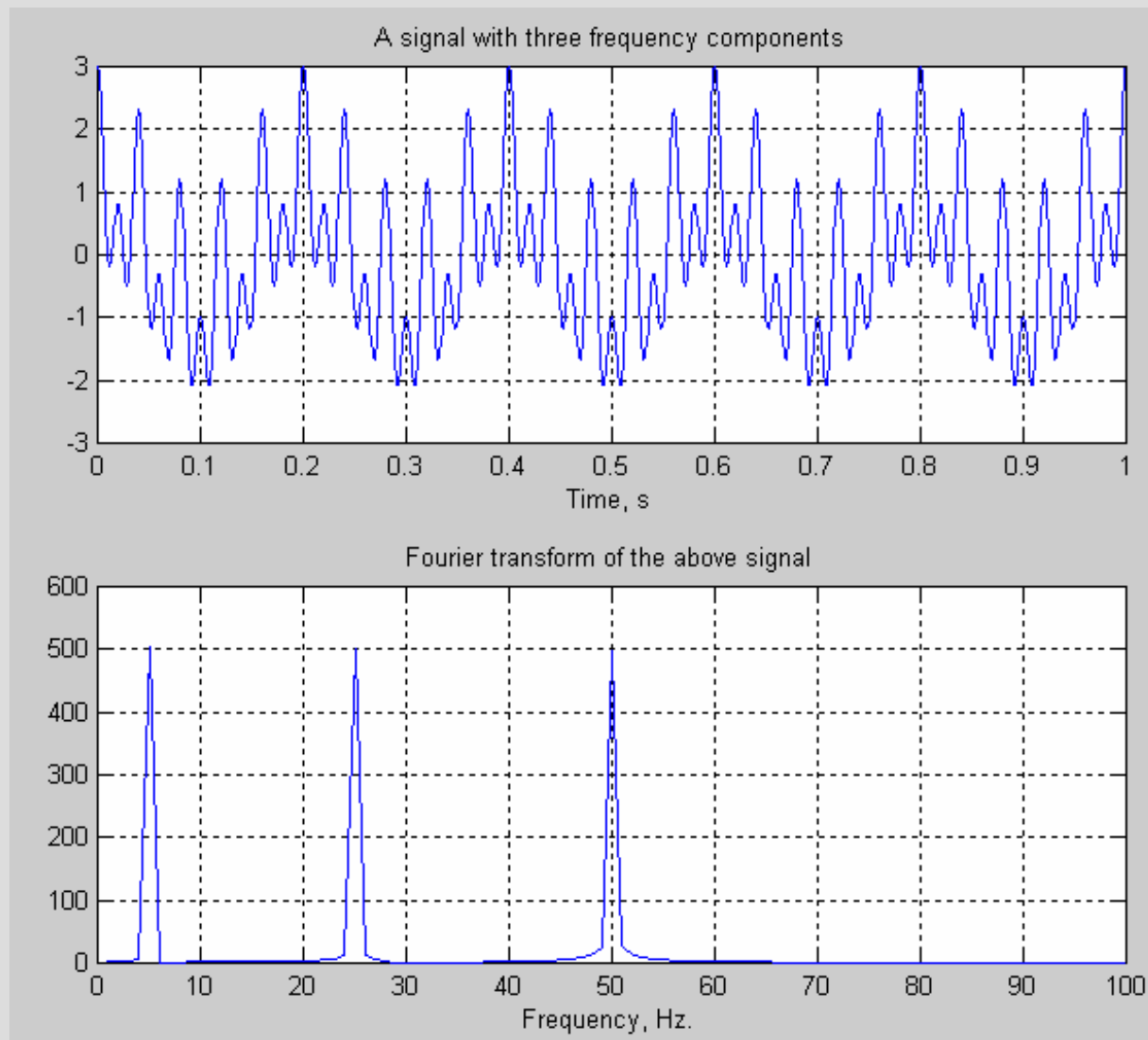
$$x_1(t) \xleftrightarrow{\mathcal{F}} X_1(\omega)$$

$$x_2(t) \xleftrightarrow{\mathcal{F}} X_2(\omega)$$

$$x_3(t) \xleftrightarrow{\mathcal{F}} X_3(\omega)$$



$$x_4(t) = \cos(2\pi \cdot 5 \cdot t) \\ + \cos(2\pi \cdot 25 \cdot t) \\ + \cos(2\pi \cdot 50 \cdot t)$$



$$x_4(t) \xleftrightarrow{\mathcal{F}} X_4(\omega)$$

Time

Frequency

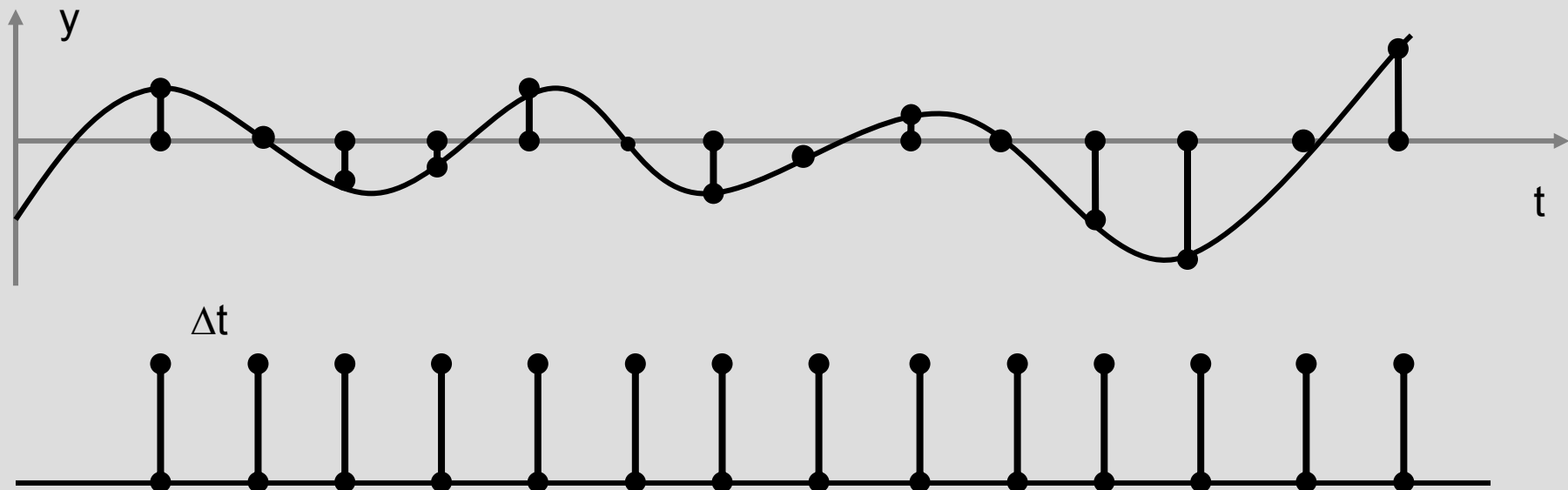
Homogeneity	$a \cdot s(t)$	$a \cdot S(k)$
Additivity	$s(t) + u(t)$	$S(k) + U(k)$
Linearity	$a \cdot s(t) + b \cdot u(t)$	$a \cdot S(k) + b \cdot U(k)$
Time reversal	$s(-t)$	$S(-k)$
Multiplication *	$s(t) \cdot u(t)$	$\sum_{m=-\infty}^{\infty} S(k-m)U(m)$
Convolution *	$\frac{1}{T} \cdot \int_0^T s(t-\bar{t}) \cdot u(\bar{t}) d\bar{t}$	$S(k) \cdot U(k)$
Time shifting	$s(t-\bar{t})$	$e^{-j \frac{2\pi k \cdot \bar{t}}{T}} \cdot S(k)$
Frequency shifting	$e^{+j \frac{2\pi m t}{T}} \cdot s(t)$	$S(k - m)$

$$F(\omega) = \int f(t) e^{-j\omega t} dt$$

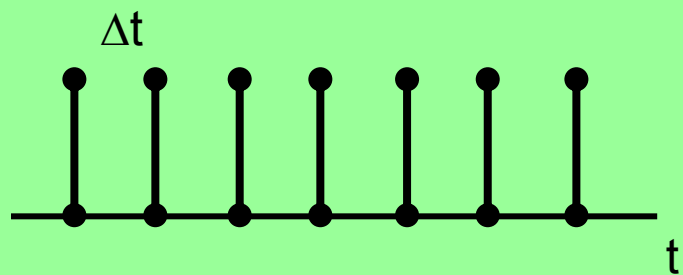
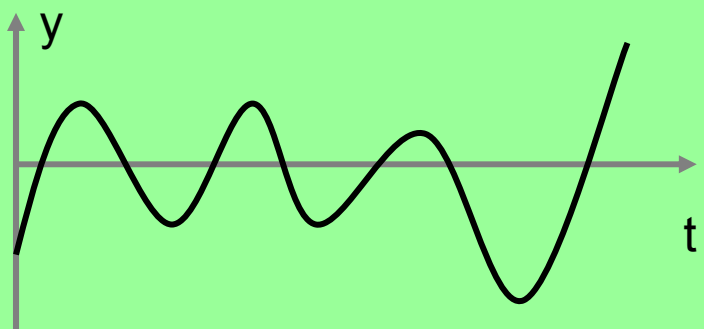
$$f(t) = \frac{1}{2\pi} \int F(\omega) e^{j\omega t} d\omega$$

Multiplication in time domain becomes Convolution in Frequency domain

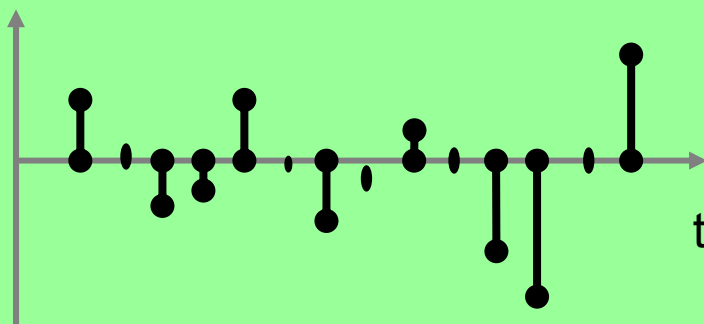
Properties of Sampling



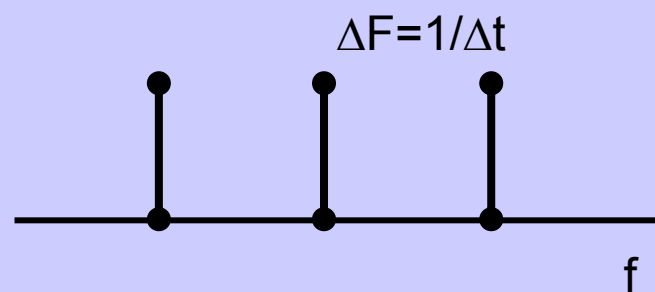
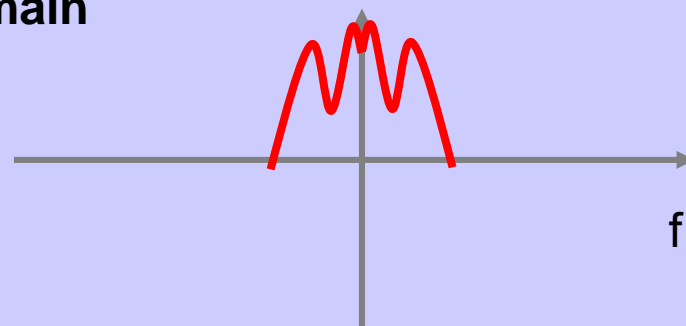
Time Domain



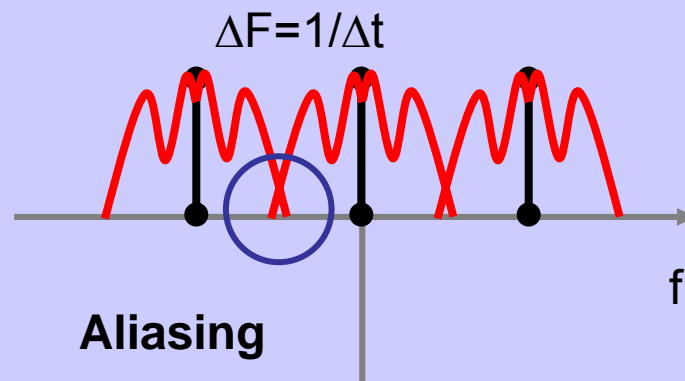
**Multi-
plication**

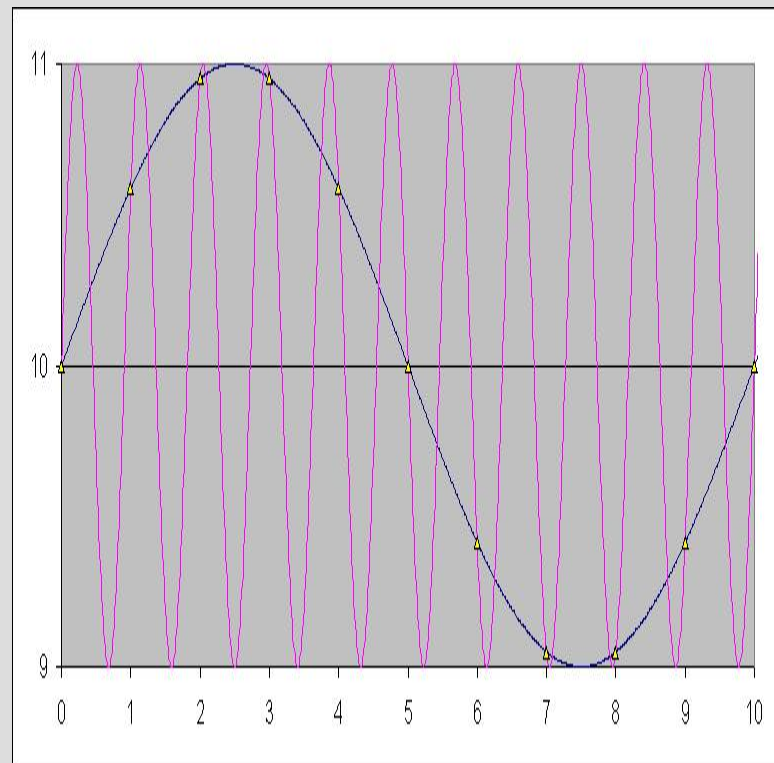
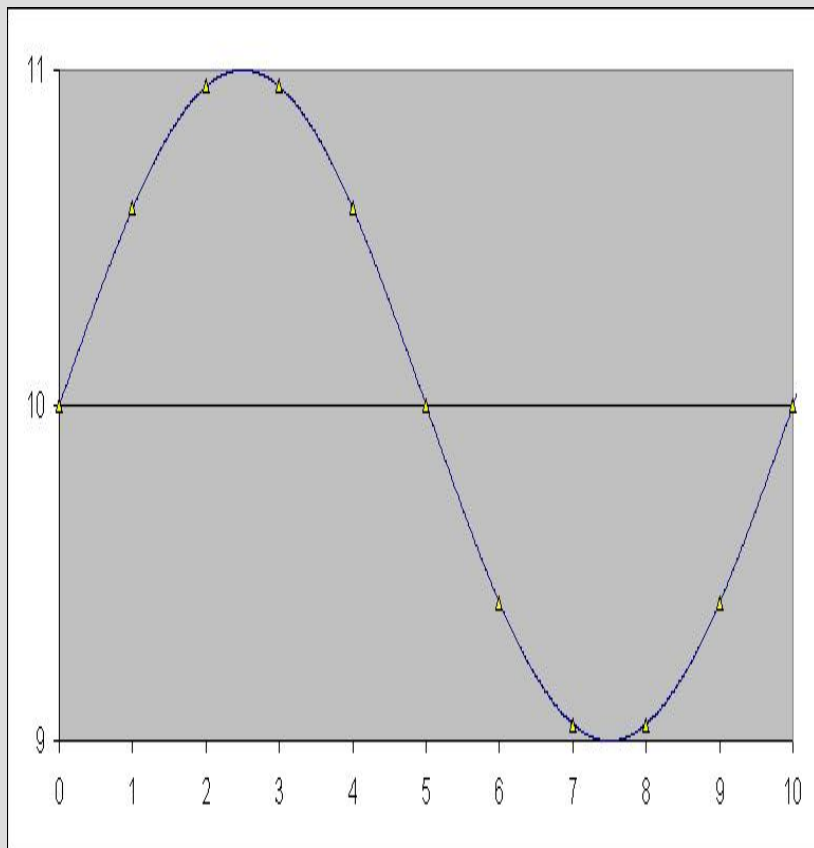


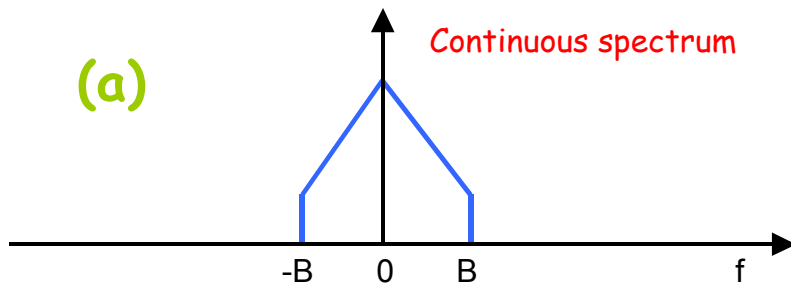
frequency Domain



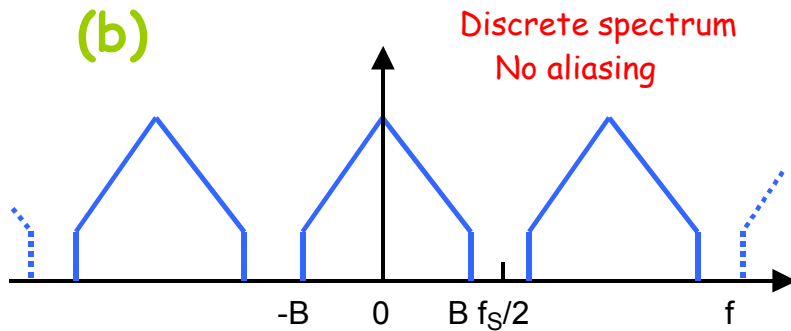
Convolution





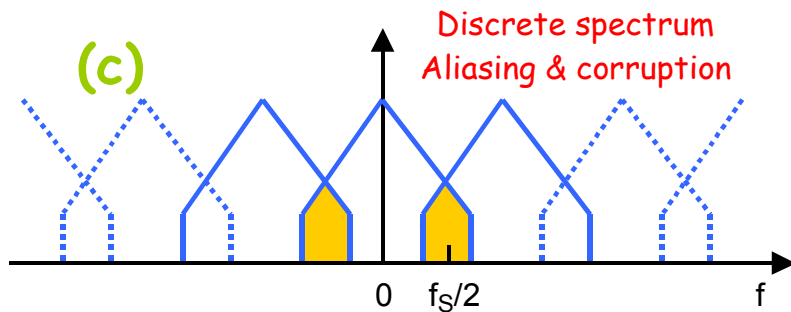


(a) Band-limited signal:
frequencies in $[-B, B]$ ($f_{MAX} = B$).



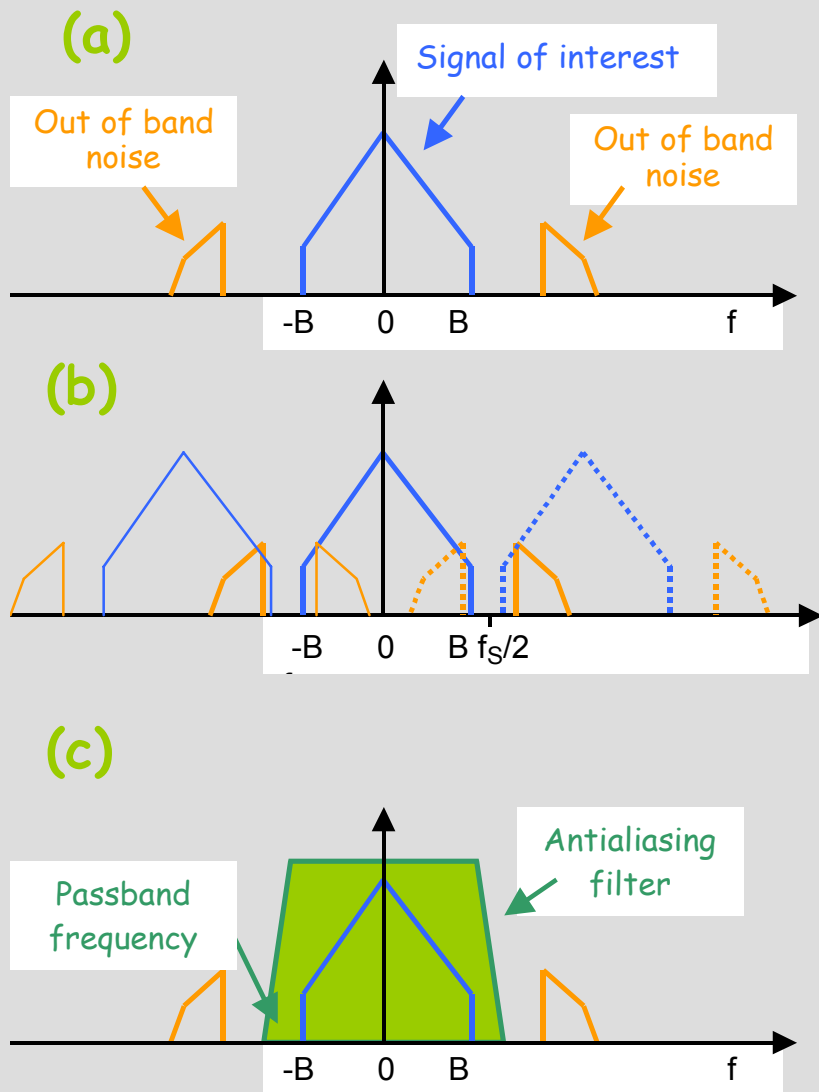
(b) Time sampling \Rightarrow frequency repetition.

$f_s > 2B \Rightarrow$ no aliasing.



(c) $f_s \leq 2B \Rightarrow$ aliasing !

Aliasing: signal ambiguity
in frequency domain



(a), (b) Out-of-band noise can alias into band of interest. Filter it before!

(c) Antialiasing filter

Passband: depends on bandwidth of interest.

Attenuation A_{MIN} : depends on

- ADC resolution (number of bits N).

$$A_{MIN, dB} \sim 6.02 N + 1.76$$

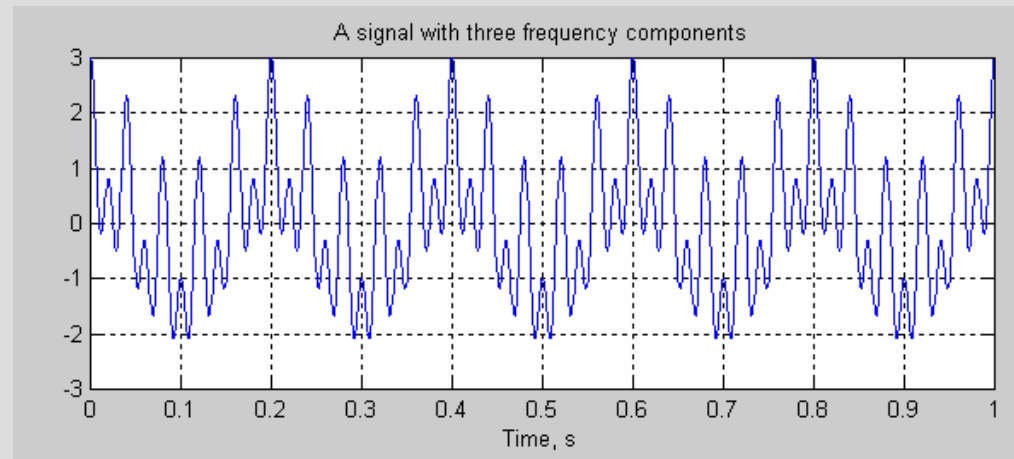
- Out-of-band noise magnitude.

Other parameters: ripple, stopband frequency...

- **FT identifies all spectral components present in the signal, however it does not provide any information regarding the temporal (time) localization of these components. Why?**

- Stationary signals' spectral characteristics do not change with time

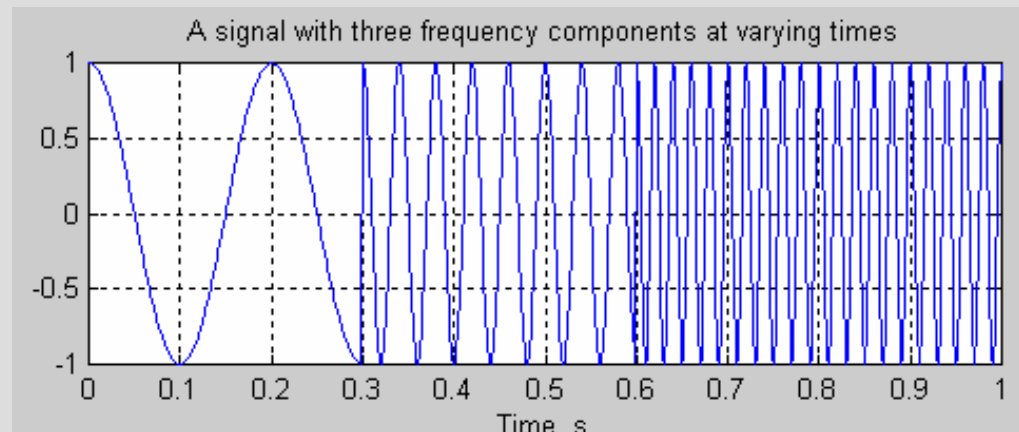
$$x_4(t) = \cos(2\pi \cdot 5 \cdot t) \\ + \cos(2\pi \cdot 25 \cdot t) \\ + \cos(2\pi \cdot 50 \cdot t)$$



- Non-stationary signals have time varying spectra

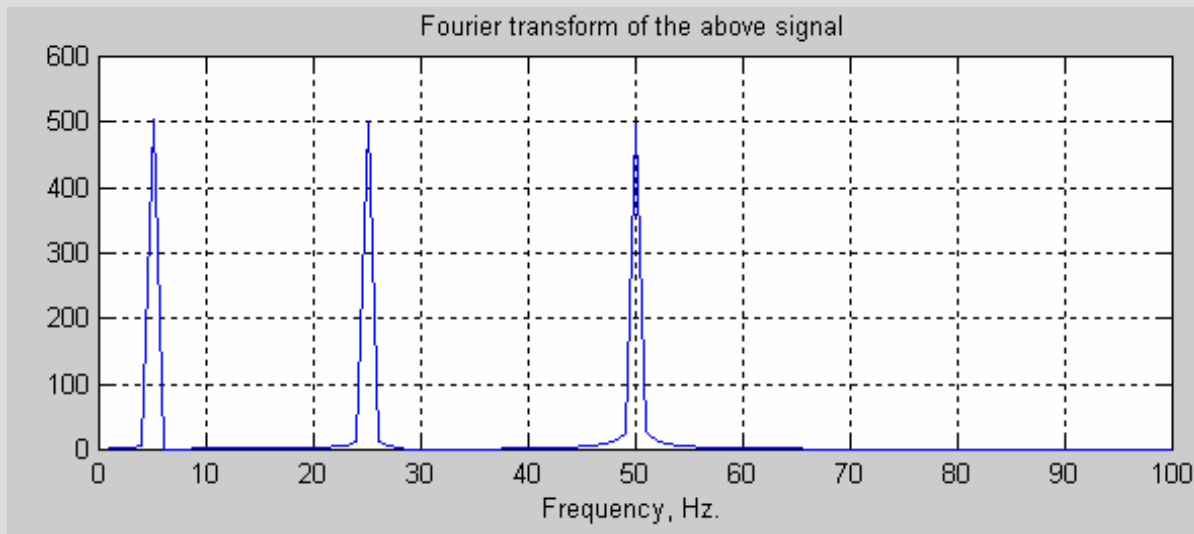
$$x_5(t) = [x_1 \oplus x_2 \oplus x_3]$$

\oplus Concatenation

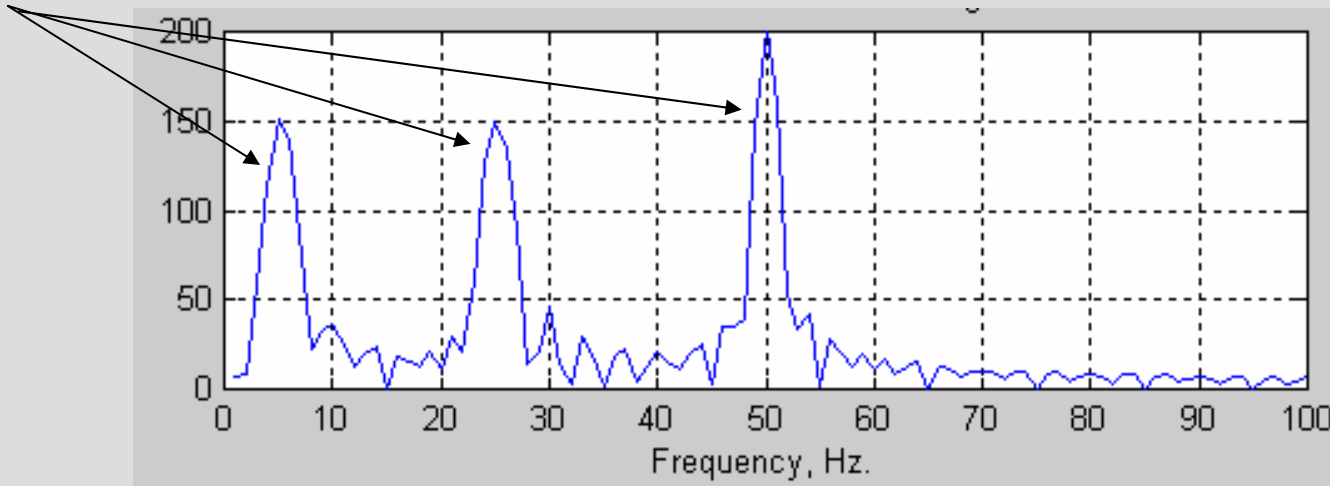


Perfect knowledge of what frequencies exist, but no information about where these frequencies are located in time

$$X_4(\omega)$$



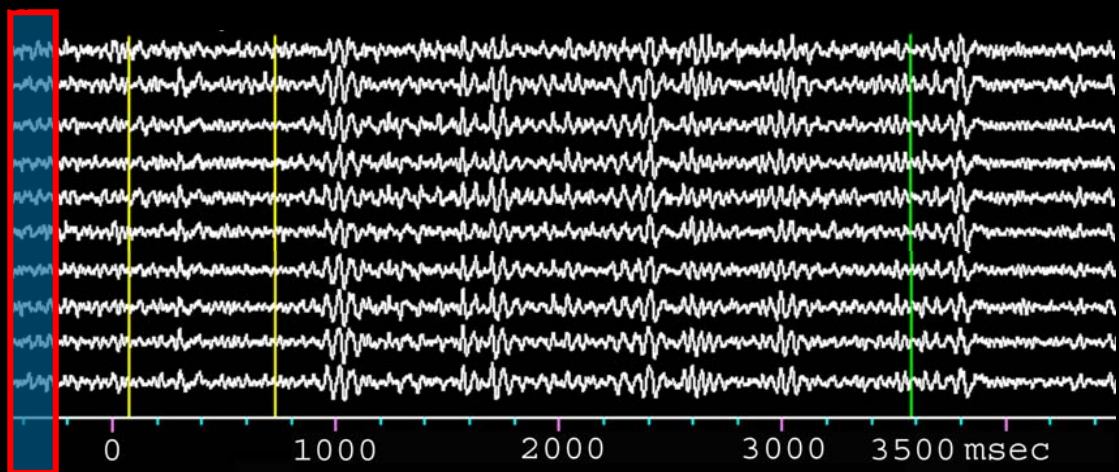
$$X_5(\omega)$$



- Sinusoids and exponentials
 - Stretch into infinity in time → no time localization
 - Instantaneous in frequency → perfect spectral localization
 - **Global** analysis does not allow analysis of non-stationary signals

- Need a **local** analysis scheme for a time-frequency representation (TFR) of nonstationary signals
 - Windowed F.T. or Short Time F.T. (STFT) : Segmenting the signal into narrow time intervals, narrow enough to be considered stationary, and then take the Fourier transform of each segment, Gabor 1946.

Sliding Window Fourier transformation of the LFP signal

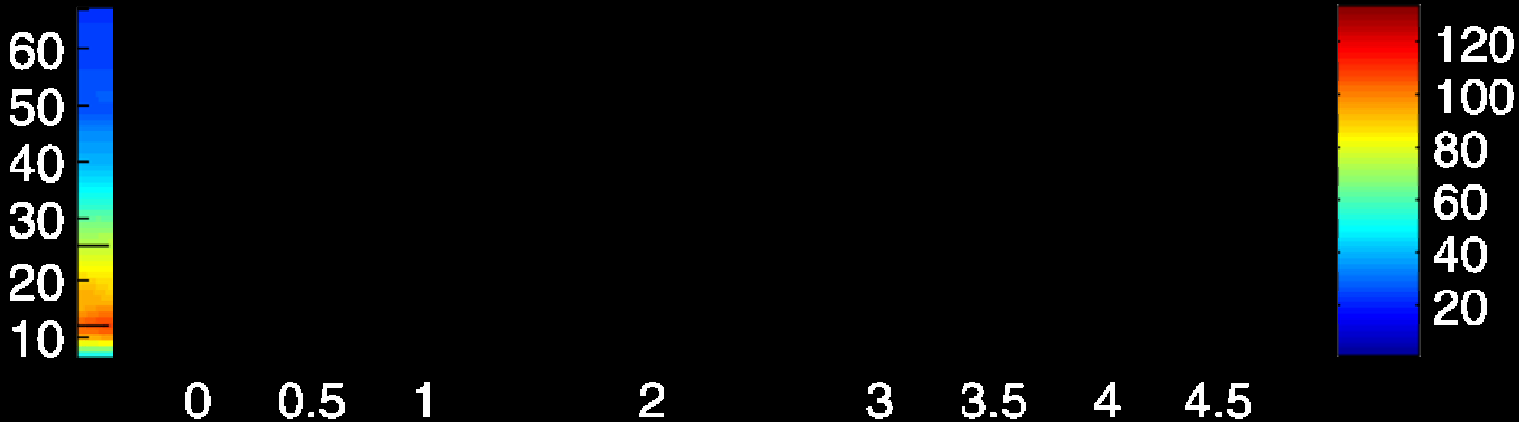



200ms

200ms

Fourier /Wavelet
Transformation

Mean Power - All Sessions Pooled

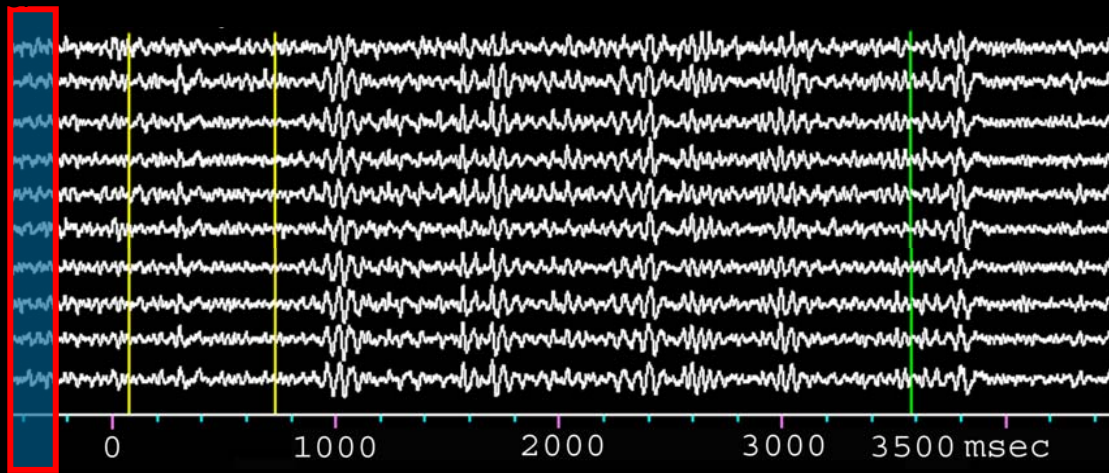


1. Choose a window function of finite length
 2. Place the window on top of the signal at $t=0$
 3. Truncate the signal using this window
 4. Compute the FT of the truncated signal, save.
 5. Incrementally slide the window to the right
 6. Go to step 3, until window reaches the end of the signal
- 

For each time location where the window is centered, we obtain a different FT

→ Hence, each FT provides the spectral information of a separate time-slice of the signal, providing simultaneous time and frequency information

Sliding Window Fourier transformation of the LFP signal

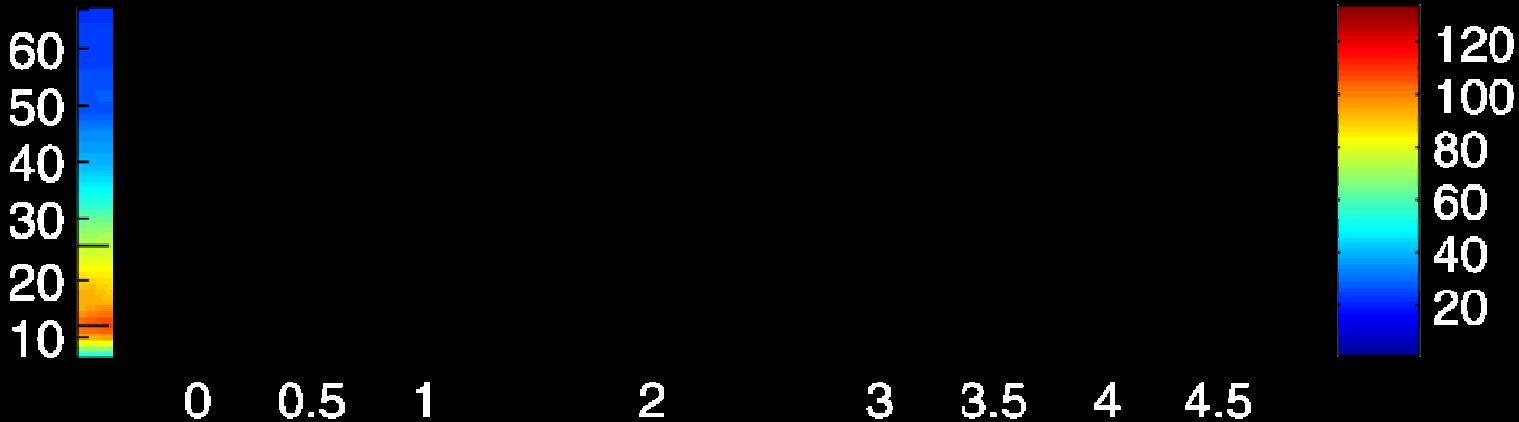


200ms

200ms

Fourier /Wavelet
Transformation

Mean Power - All Sessions Pooled



Time parameter

Frequency parameter

Signal to be analyzed

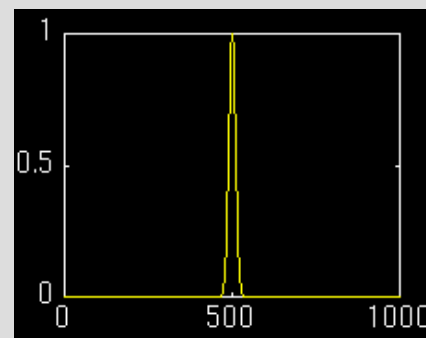
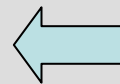
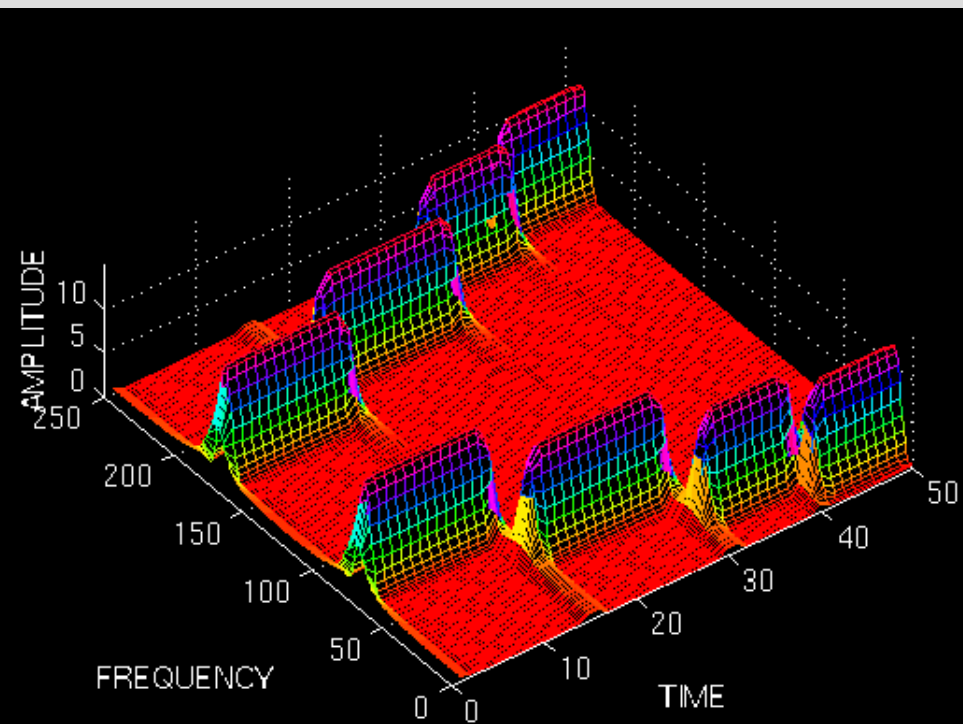
FT Kernel (basis function)

$$STFT_x^\omega(t', \omega) = \int_t [x(t) \cdot W(t - t')] \cdot e^{-j\omega t} dt$$

STFT of signal $x(t)$:
Computed for each
window centered at $t=t'$

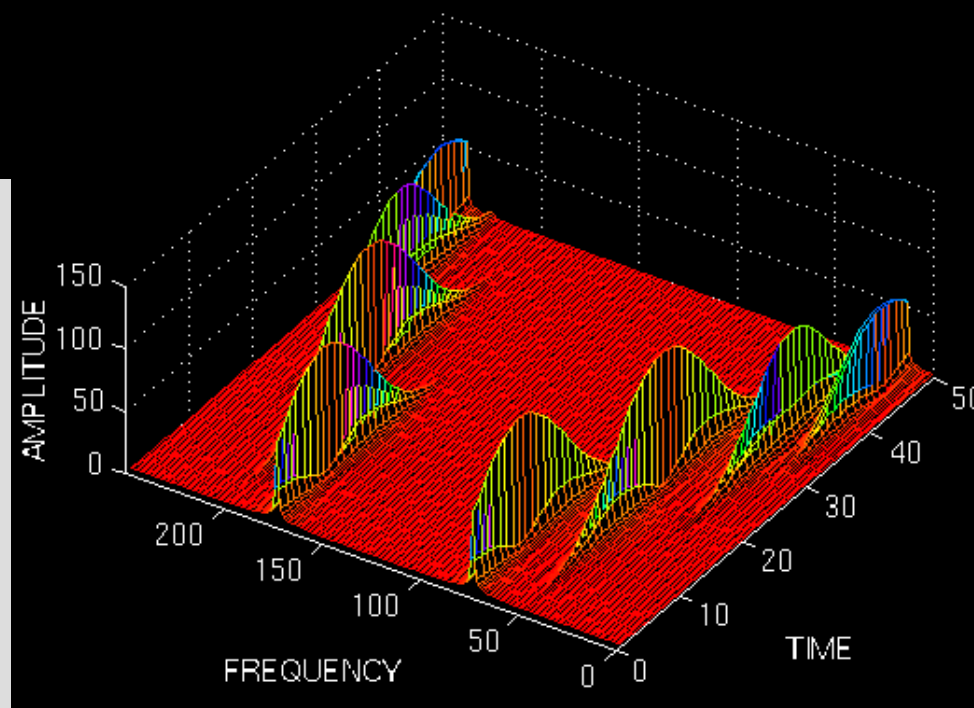
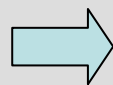
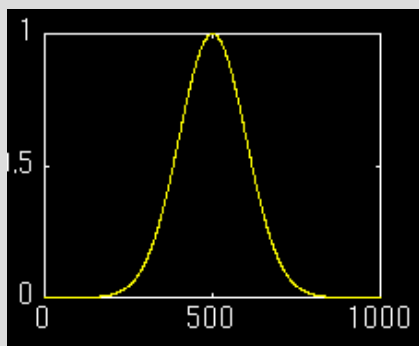
Windowing
function

Windowing function
centered at $t=t'$



Sharp window in time

broader window in time



- STFT provides the time information by computing a different FTs for consecutive time intervals, and then putting them together
 - Time-Frequency Representation (TFR)
 - Maps 1-D time domain signals to 2-D time-frequency signals

- Consecutive time intervals of the signal are obtained by truncating the signal using a sliding windowing function

- How to choose the windowing function?
 - Wider window require less time steps \rightarrow low time resolution
 - Also, window should be narrow enough to make sure that the portion of the signal falling within the window is stationary

$$STFT_x^\omega(t', \omega) = \int_t [x(t) \cdot W(t - t')] \cdot e^{-j\omega t} dt$$

Two extreme cases:

- $W(t)$ infinitely long: $W(t) = 1$

→ STFT turns into FT, providing excellent frequency information (good frequency resolution), but no time information

$$\Delta t \cdot \Delta f \geq \frac{1}{4\pi}$$

Time resolution:

How well two spikes in time can be separated from each other in the transform domain

Frequency resolution:

How well two spectral components can be separated from each other in the transform domain

Both time and frequency resolutions cannot be arbitrarily high!!!

→ We cannot precisely know at what time instance a frequency component is located. We can only know what *interval of frequencies* are present in which *time intervals*

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- Overcomes the preset resolution problem of the STFT by using a variable length window
- Analysis windows of different lengths are used for different frequencies:
 - Analysis of high frequencies → Use narrower windows for better time resolution
 - Analysis of low frequencies → Use wider windows for better frequency resolution
- This works well, if the signal to be analyzed mainly consists of slowly varying characteristics with occasional short high frequency bursts.
- Heisenberg principle still holds!!!
- The function used to window the signal is called *the wavelet*

Translation parameter, measure of time Scale parameter, measure of frequency A normalization constant Signal to be analyzed

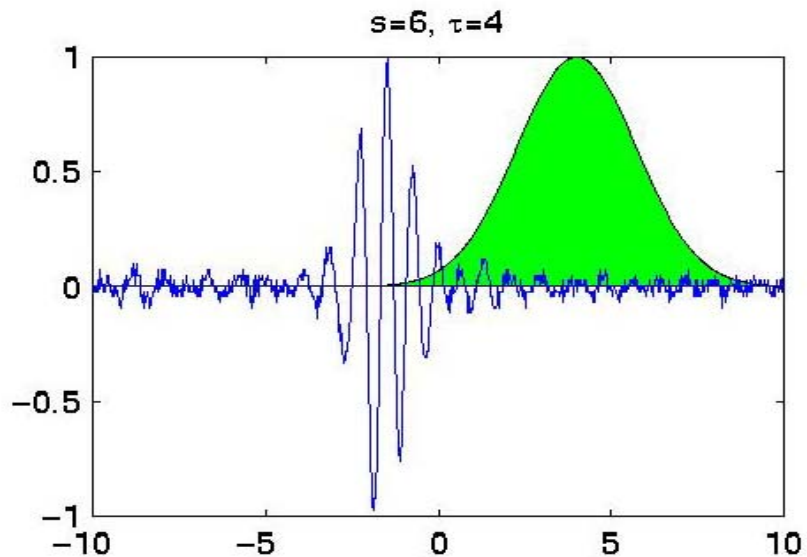
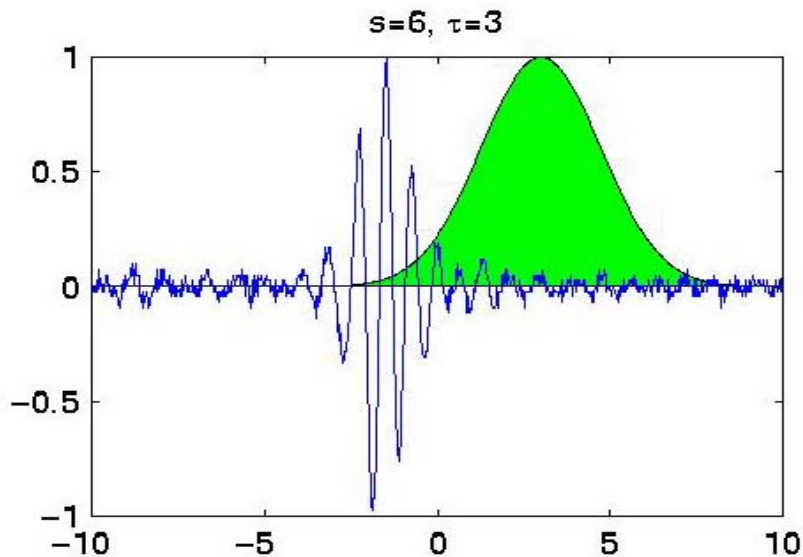
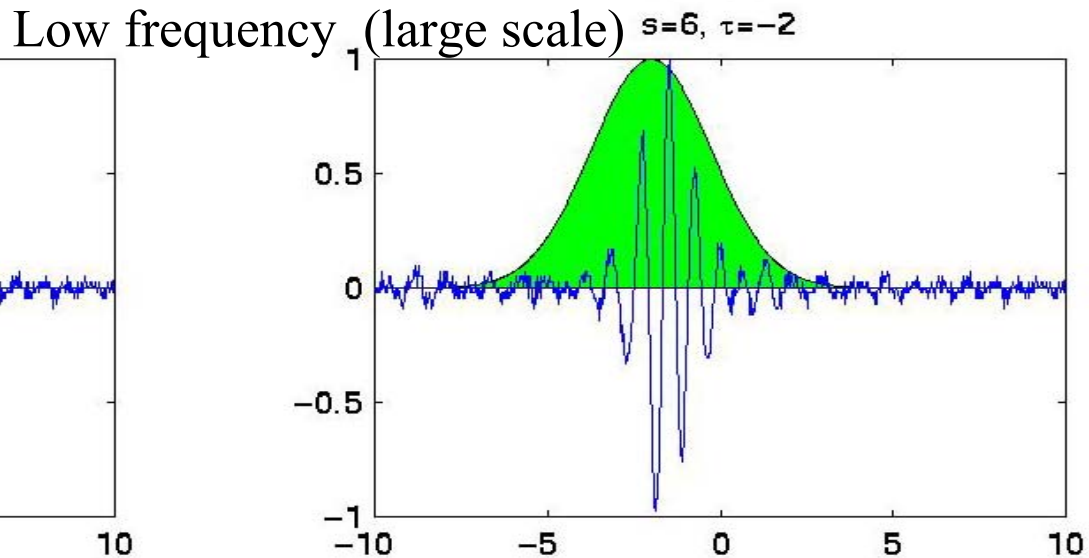
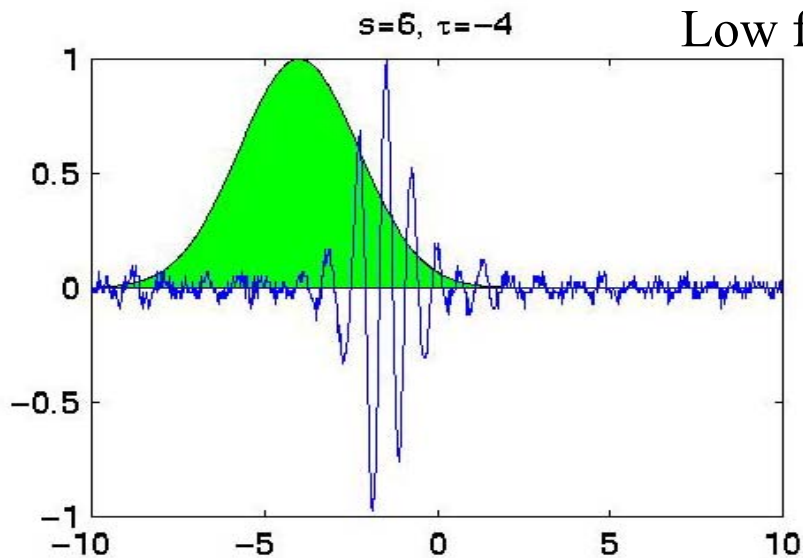
$$CWT_x^\psi(\tau, s) = \Psi_x^\psi(\tau, s) = \frac{1}{\sqrt{|s|}} \int_t x(t) \psi^*\left(\frac{t - \tau}{s}\right) dt$$

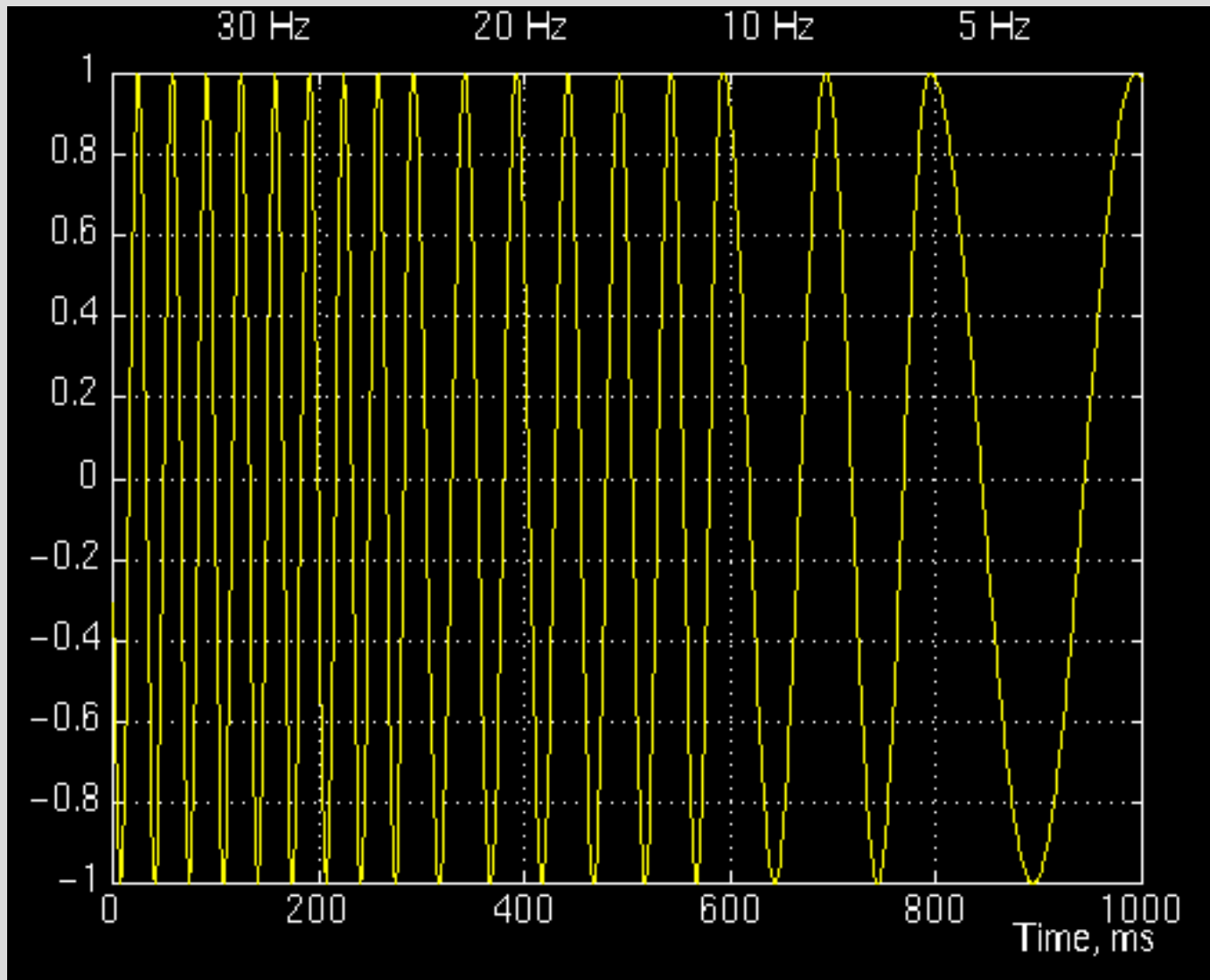
Continuous wavelet transform of the signal $x(t)$ using the analysis wavelet $\psi(\cdot)$

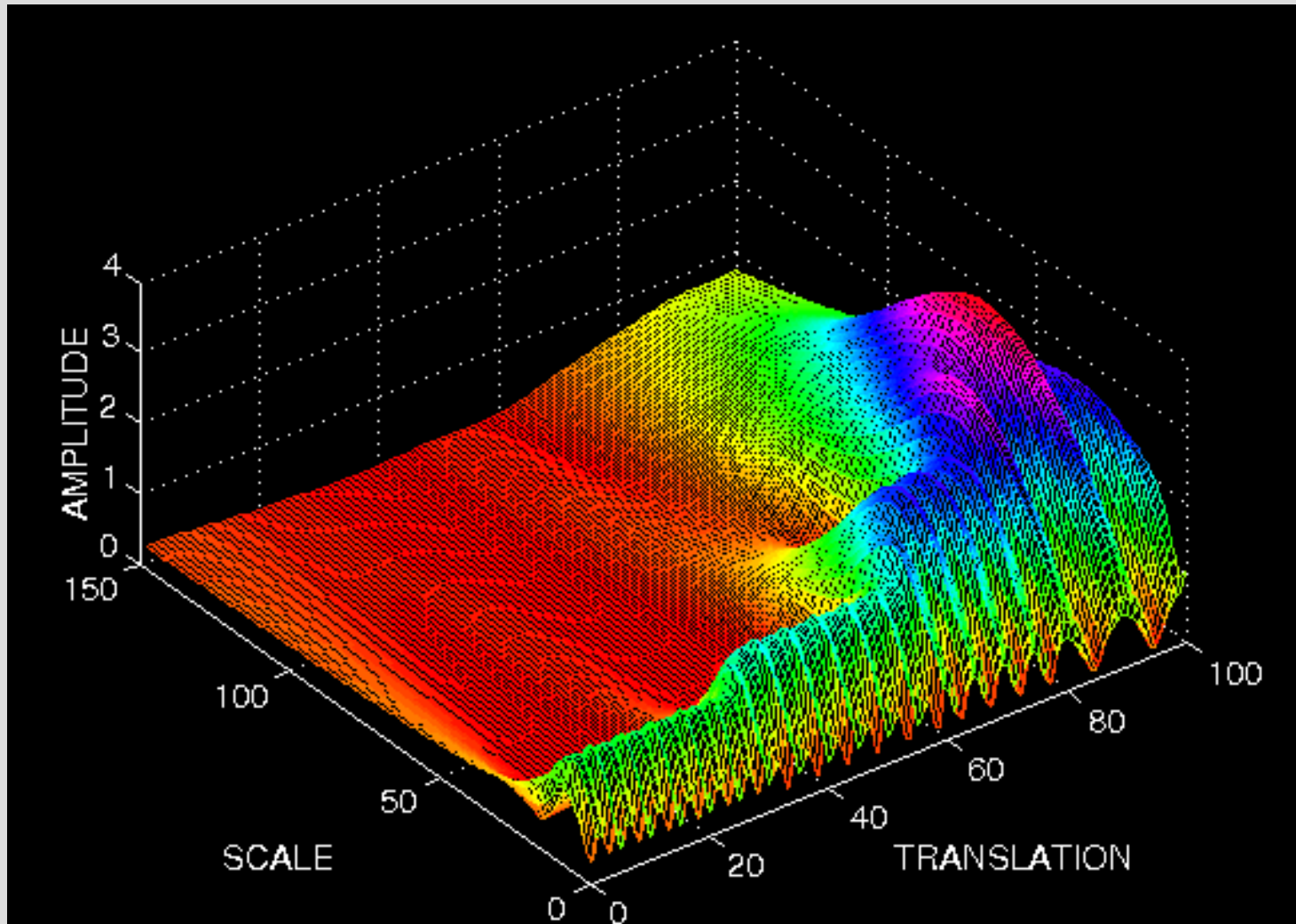
The mother wavelet. All kernels are obtained by translating (shifting) and/or scaling the mother wavelet

Scale = 1/frequency

$$CWT_x^\psi(\tau, s) = \Psi_x^\psi(\tau, s) = \frac{1}{\sqrt{|s|}} \int_t x(t) \psi^* \left(\frac{t-\tau}{s} \right) dt$$

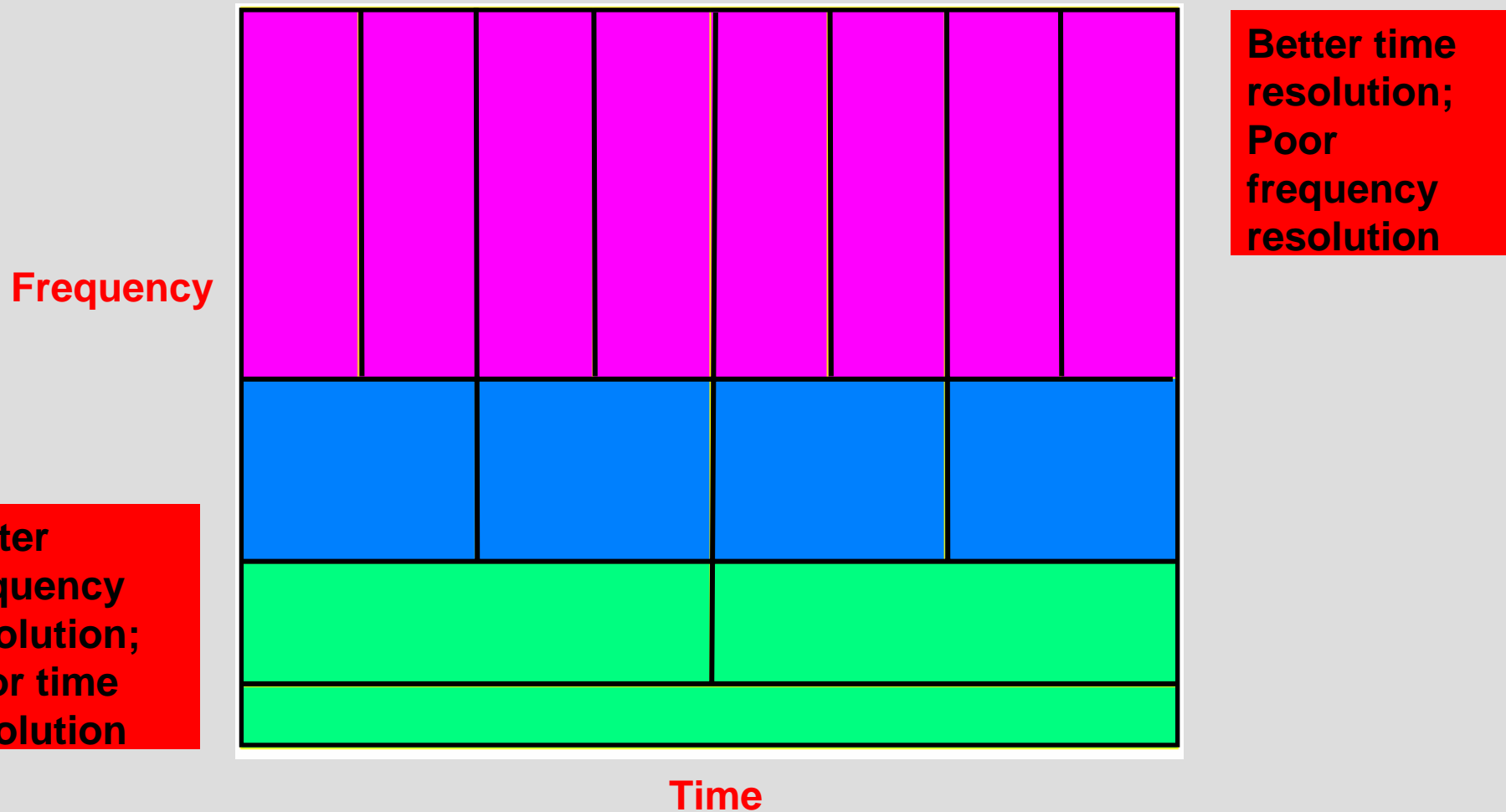






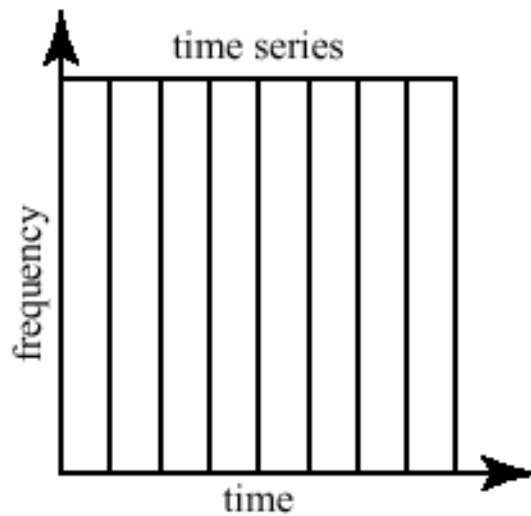
$$\text{CWT}_x^\psi(\tau, s) = \Psi_x^\psi(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \bullet \psi^* \left(\frac{t - \tau}{s} \right) dt$$

- Step 1:** The wavelet is placed at the beginning of the signal, and set $s=1$ (the most compressed wavelet);
- Step 2:** The wavelet function at scale “1” is multiplied by the signal, and integrated over all times; then multiplied by $1/\sqrt{|s|}$
- Step 3:** Shift the wavelet to $t = \tau$, and get the transform value at $t = \tau$ and $s=1$;
- Step 4:** Repeat the procedure until the wavelet reaches the end of the signal;
- Step 5:** Scale s is increased by a sufficiently small value, the above procedure is repeated for all s ;
- Step 6:** Each computation for a given s fills the single row of the time-scale plane;
- Step 7:** CWT is obtained if all s are calculated.

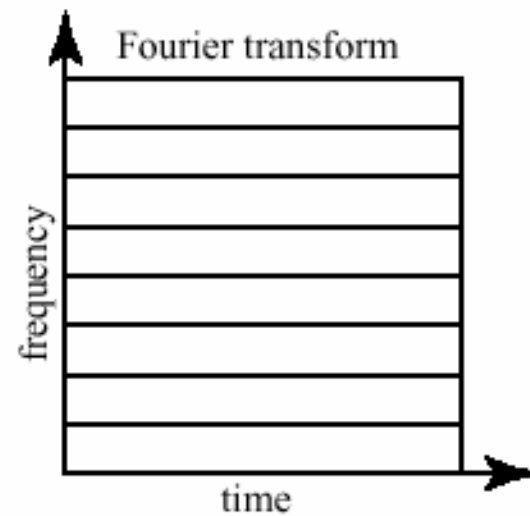


Regardless of the dimensions of the boxes, the areas of all boxes, both in STFT and WT, are the same and determined by **Heisenberg's inequality** .

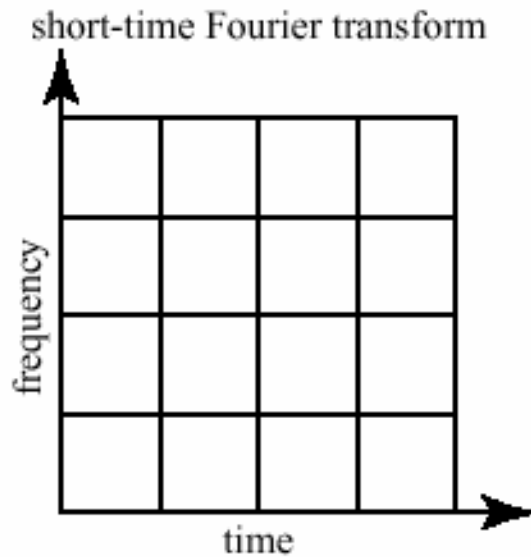
Full time but no
frequency info



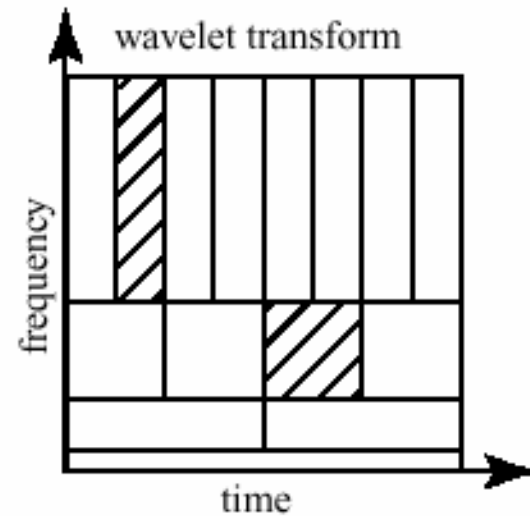
Full frequency but
no time info



frequency and
time resolution
always the same



Higher time
resolution for
high frequencies



1. Concepts in time series Analysis
2. Basic Statistics
3. Auto Correlation
4. Auto Regressive Modeling
5. Fourier Transformation
6. Wavelet Transformation

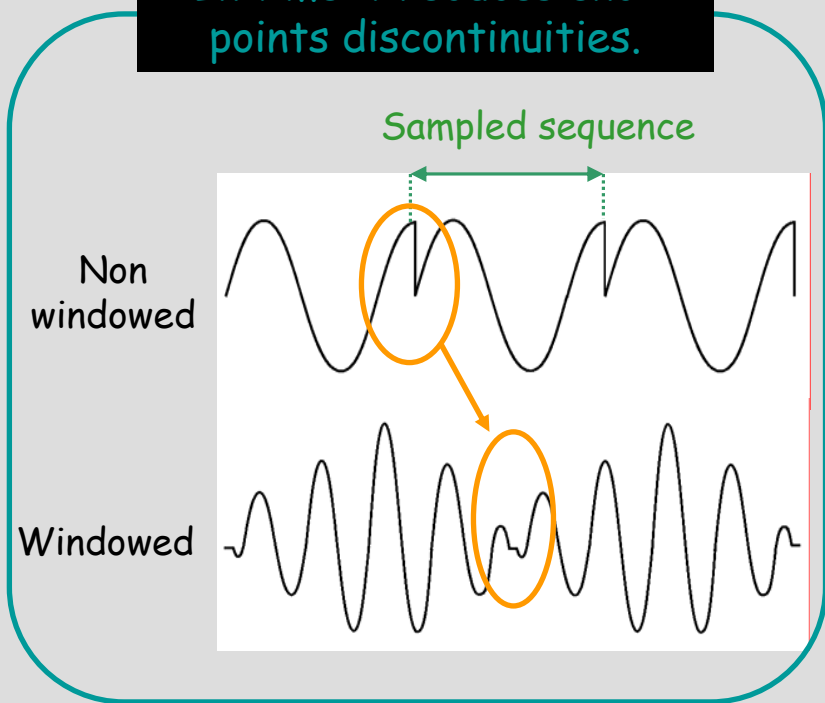
The end

	$f(t) = \int_{-\infty}^{\infty} F_F(i\omega) e^{i\omega t} \frac{d\omega}{2\pi}$	$F_F(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$	
	$\text{rect} \frac{t}{T} = \begin{cases} 1 & (t < T/2) \\ 0 & (t > T/2) \end{cases}$	$T \text{sinc} \frac{\omega T}{2\pi} \equiv T \frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}}$	
	$\text{sinc} \frac{t}{T} \equiv \frac{\sin \frac{\pi t}{T}}{\frac{\pi t}{T}}$	$T \text{rect} \frac{\omega T}{2\pi} = \begin{cases} 0 & (\omega < \frac{\pi}{T}) \\ T & (\omega > \frac{\pi}{T}) \end{cases}$	
	$\begin{cases} 1 - \frac{ t }{T} & (t < T) \\ 0 & (t \geq T) \end{cases}$	$T \text{sinc}^2 \frac{\omega T}{2\pi} \equiv T \left(\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2$	
	$e^{-\frac{ t }{T}}$	$\frac{2T}{(\omega T)^2 + 1}$	
	$e^{-\frac{1}{2} \left(\frac{t}{T}\right)^2}$	$\sqrt{2\pi} T e^{-\frac{1}{2} (\omega T)^2}$	

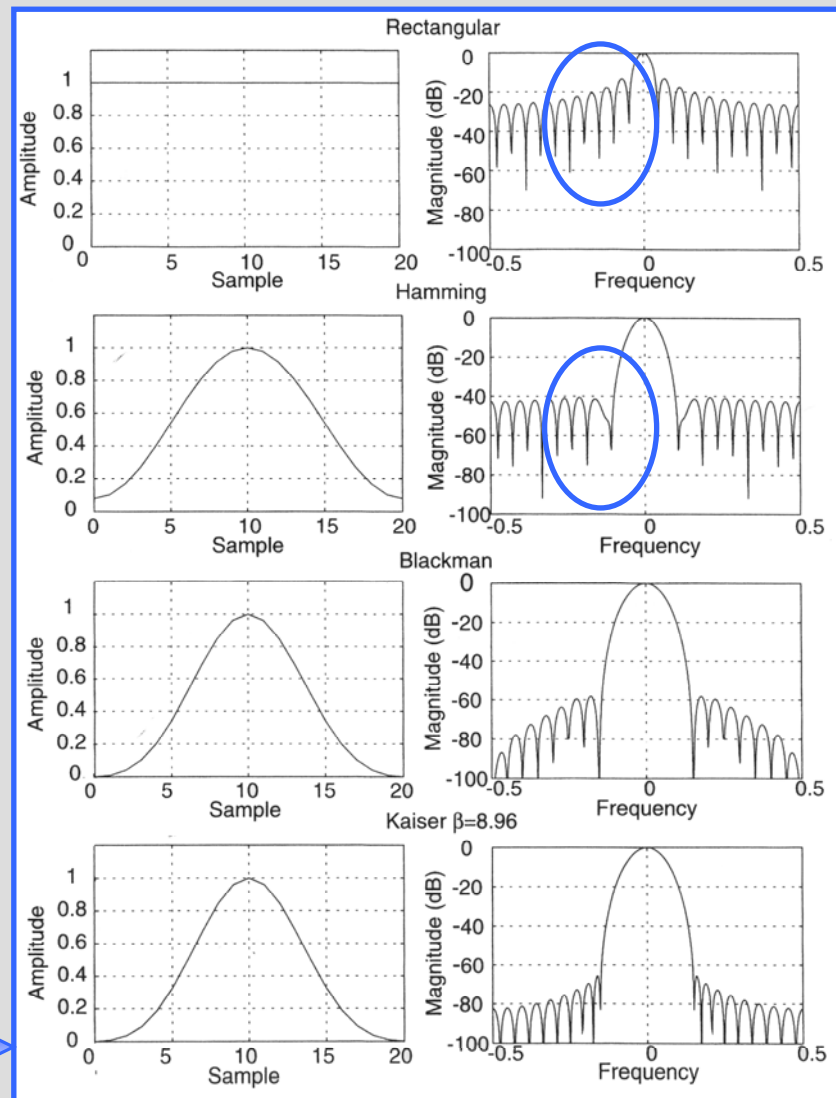
	$f(t) = \int_{-\infty}^{\infty} F_F(i\omega) e^{i\omega t} \frac{d\omega}{2\pi}$	$F_F(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$	
	$\delta(t - T)$	$e^{-i\omega T}$	(Complex)
	$\cos \omega_0 t$	$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
	$\sin \omega_0 t$	$\frac{\pi}{i} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	(Imaginary)
	$\sum_{k=-\infty}^{\infty} \delta(t - kT)$ $\equiv \frac{1}{T} \sum_{j=-\infty}^{\infty} e^{2\pi i j \frac{t}{T}}$	$\frac{2\pi}{T} \sum_{j=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi j}{T}\right)$ $\equiv \sum_{k=-\infty}^{\infty} e^{ik - T}$	

Windowing reduces leakage by minimising sidelobes magnitude.

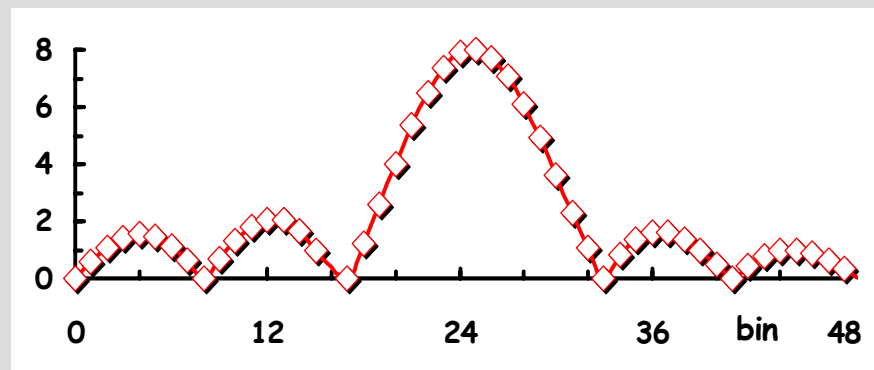
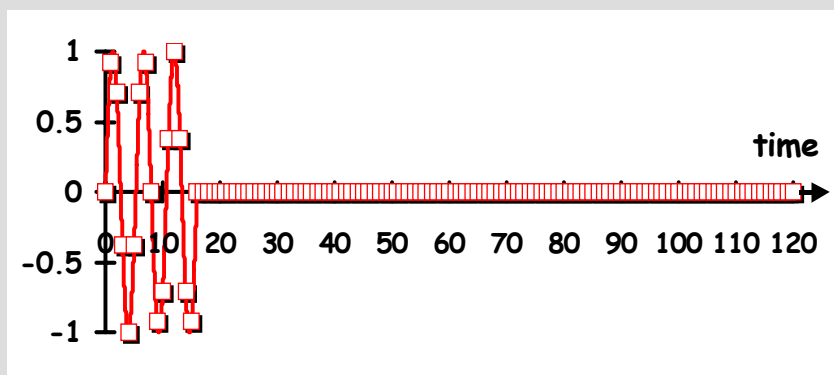
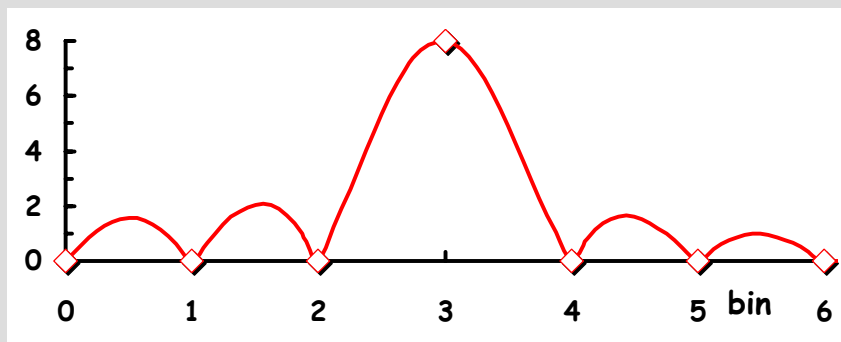
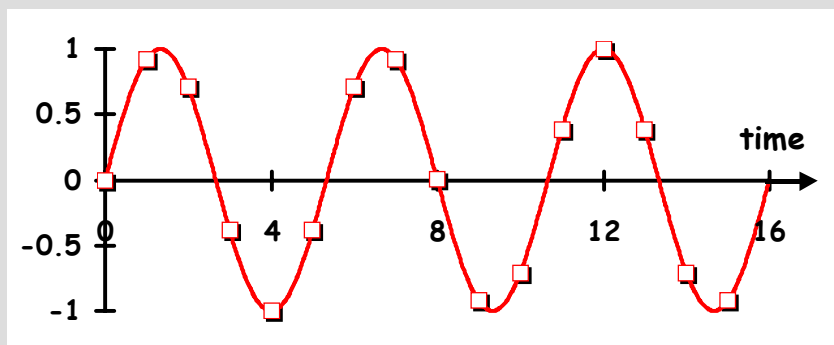
In time it reduces end-points discontinuities.



Some window functions



Improves DFT frequency inter-sampling spacing ("resolution").



After padding bins @ frequencies $f_m = \frac{m \cdot f_s}{N_S + L}$ $N_S = \text{original samples}, L = \text{padded}.$

DFT spectral resolution

Capability to distinguish two closely-spaced frequencies: not improved by zero-padding!

Frequency inter-sampling spacing: increased by zero-padding (DFT “frequency span” unchanged due to same sampling frequency)

- Zero-padding in frequency domain increases sampling rate in time domain. Note: it works only if sampling theorem satisfied!
- Additional reason for zero-padding: to reach a power-of-two input samples number (see FFT).

NOTE

Apply zero-padding after windowing (if any)! Otherwise stuffed zeros will partially distort window function.



WAVELET FAMILY PROPERTIES

Property	morl	mexh	meyr	haar	dbN	symN	coifN	biorNr.Nd	rbioNr.Nd	gaus	dmey	cgau	cmor	fbsp	shan
Crude	•	•								•		•	•	•	•
Infinitely regular	•	•	•							•		•	•	•	•
Arbitrary regularity					•	•	•	•	•						
Compactly supported orthogonal				•	•	•	•								
Compactly supported biorthogonal								•	•						
Symmetry	•	•	•	•				•	•	•	•	•	•	•	•
Asymmetry					•										
Near symmetry						•	•								
Arbitrary number of vanishing moments					•	•	•	•	•						
Vanishing moments for ϕ							•								
Existence of ϕ			•	•	•	•	•	•	•						
Orthogonal analysis			•	•	•	•	•								
Biorthogonal analysis			•	•	•	•	•	•	•						
Exact reconstruction	\mathbb{R}	•	•	•	•	•	•	•	•	•	\mathbb{R}	•	•	•	•
FIR filters				•	•	•	•	•	•		•				
Continuous transform	•	•	•	•	•	•	•	•	•	•					
Discrete transform			•	•	•	•	•	•	•		•				
Fast algorithm				•	•	•	•	•	•		•				
Explicit expression	•	•		•				For splines	For splines	•		•	•	•	•
Complex valued												•	•	•	•
Complex continuous transform												•	•	•	•
FIR-based approximation											•				